ANALEMMAS

Or how to build dials with the method in use in the epoch of the ancient Greeks and Romans.

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ANALEMMAS
or
The ANALEMMAS of VITRUVIUS and PTOLEMY

Introduction

I tried to do a bibliography on Gnomonics, gathering only the titles listed in recent books. I found about forty - no more; but it is fairly normal for a bibliography compiled by a hobbyist to be rather incomplete. Yet, even from such a summarized list, I could draw a conclusion: One problem, which they all considered – that of projecting straight lines and circles onto a surface - must have an exceptional charm to be the object of so much interest.

Probably the charm is not in the problem, but in some other thing.

I then began to read the books on the list: some old, some more recent. Not all, please! I came to a rather interesting conclusion: after Regiomontanus and the work of Münster (1531) and of Clavius (1580), the others are "imitations", only adapting earlier solutions to progress of some area of mathematics.

E.g., the diffusion of sine tables, the "discovery" of logarithms and trigonometry take the lion's share; but virtually nothing is original. Also because after all it is only one problem, and resolved already in an exhaustive way, it could be turned to the systems of hour subdivision, to the zodiac signs, to mean time, *et cetera*, but it stays substantially always the same. Yet Clavius devotes a great part of his most important book on the mathematical development of the matter.

I admit a certain type of disappointment: I expected, above all from the ancient texts, an ample illustration of the graphic methods (I admit it: they fascinate me, more than the analytical methods). But they are almost always omitted: the populariser Bédos de Celles tells it straight - that the graphic solutions are not accurate. He explains them, but reluctantly, and he does not apply them. Only the 19th Century texts take the graphic approach and give dignity to it.

Then I understood: geometric speculation, appreciated by the Greeks, had a moment of eclipse in the XVIth, XVIIth, and XVIIIth centuries, which produced personalities like Pascal and Desargues. Those centuries developed the analytical search (select any name, at random: Descartes, Briggs, Napier, Leibnitz, Newton) so much that the treatise on the projective properties of figures, by Poncelet (and we are already in 1822), was greeted rather coldly by the Academicians. Only the parallel development of engineering, with the need for graphic illustration of the work, gave life to geometry.

Finally I found: one, Muzio Oddi (first edition of his work, 1614, Milan). He makes up the Analemma, adapts it, develops and illustrates it, shows the elegance and the simplicity of it.

I had not yet discovered Commandino and his Latin translation of the "De Analemmate" of Ptolemy.

Sharon Gibbs in her now famous *Greek and Roman Sundials* devotes a chapter to the Analemmas, that of Vitruvius and that of Ptolemy: she quotes the texts of Drecker and Bilfinger as discoverers and popularisers of the versatility of the Analemmas. And she barely mentions Commandino. I think that she never saw the works of Muzio Oddi.

Ptolemy lived in the second century after Christ; today, in the Italian schools, he is considered particularly as contrasted to Galileo and Copernicus. His astronomic theory was logical and coherent with the available data of the epoch, so much so that it stood around 1300 years after him, almost up to the discovery of the telescope (but still in the entire 17th century the heliocentric theory was rejected by many, and Copernicus, Kepler and Galileo were snubbed. Also Cassini never said that the Earth was to turn around the Sun). Ptolemy's work is called *Al Magist*, that is, *the superlative*.

A secondary chapter of his Treatise, the Analemma, is the key to the ancient dials; it also was the "secret of manufacture" of the builders before the 16th century, jealously handed down, but not too widely divulged so as not to lose the monopoly.

And then, from the 16th century on, it was practically forgotten: or rather, only a small part was in frequent use and perhaps with imperfect intelligence, without explanation (or without knowledge) of its origin.

Yet, from the point of view of geometric speculation, the Analemma is a small masterpiece: it reduces directions and points at infinity to points of a spherical surface - tractable graphically with much simplicity, anticipating (but the others are late) the theories of projective geometry; it plays with bundles and planes of stars, with imagination and lightness, but above all it proves to be the fruit of only one idea, and is extremely compact and universal.

We could refer to the general theory of the astrolabe (the projection of the sphere onto the plane) but perhaps there is no need.

To anyone who is patient enough to read it, this is a modest essay on the method of construction of sundials, based only on the use of the Analemma.

My accomplices: Muzio Oddi and Federico Commandino.

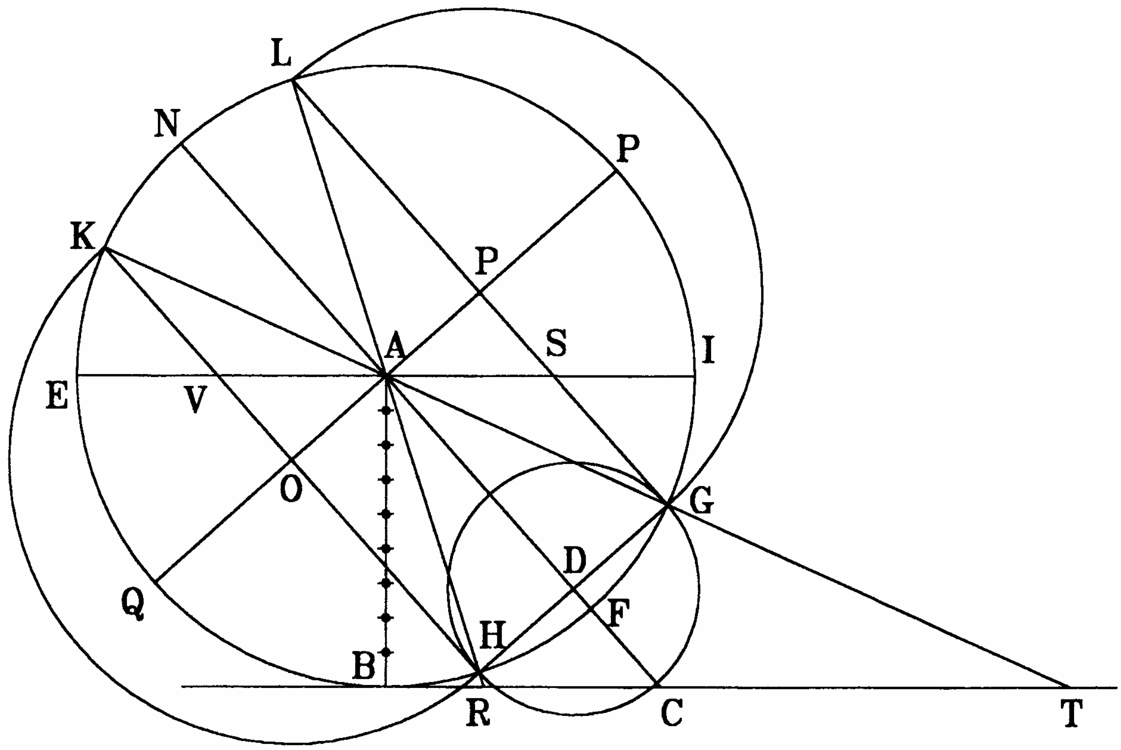
A small note: almost none of the authors of the 16th and 17th centuries quotes sources; Muzio Oddi is one of the few exceptions. He puts the source in the margin near each deserving point. And at the end of his life he expressed his disappointment, because a fellow, FULIGATTI, who had read the manuscript of his second treatise, preceded him in the publication, robbing the ideas.

(I know the work of Father Giulio Fuligatti S.J.: Degli Oriuoli da Sole, Ferrara, 1616. The general character is very similar to the works of Muzio Oddi.)

Biella, 1993-2002

From Vitruvius

Book Nine Chapter VII. The Analemma



The ANALEMMA according to Vitruvius

We will confine ourselves to explaining with clarity the decrease and increase in the length of days from one month to another. When the sun at the time of the equinox is in Aries or in Libra, it casts a shadow at the latitude of Rome equal to 8/9 of the length of the gnomon. However, at Athens, the shadow is 3/4 of the gnomon; at Rhodes 5/7; at Taranto 9/11; at Alexandria 3/5; and at all other places, the equinoctial shadow of the gnomon has different lengths, according to the natural difference (of latitude).

Therefore, if an individual desires to trace a dial, he will determine the equinoctial shadow: and if, as in Rome, it is equal to 8/9 of the gnomon, he will trace a line on a plane and in its middle point raise a perpendicular that is called a "gnomon". Then, beginning from the line traced in the plane, the length of the gnomon will be divided in nine parts with the compass and, let the point of the last of these parts be the center of the figure, marked with A. Then, extending the compasses from this center up to the base line, to the intersection point B, a circle will be traced that we will call "Meridian."

Then we take a measure equal to eight of the nine parts into which the AB distance is divided, from the base line to the center of the gnomon and mark it again on the base line, thus fixing the C point.

This will be the length of the shadow of the gnomon at the equinox. From point C we trace a line passing through the center A, and it will represent the sun ray at the equinox.

Then opening the compass to the distance from the center to the base line we will find on either side the circumference points E and I, most distant from the gnomon, respectively equidistant from the center, the one to the left, the other to the right. We will then join them by tracing a line passing through the center of the circle, that will divide the circle in two equal semicircles. This diameter is called by the mathematicians the "horizon".

Then take a fifteenth part of the whole circumference and point the compass in the point F, the intersection between the circumference and the equinoctial ray, identifying to its right and to its left the points G and H. Beginning from these and passing through the center, we draw two lines until they meet the base line in T and R.

We will thus have the representation of two solar rays, one in the winter and the other in the summer. Opposite the letter E we have the letter I, where the straight line passing through the center A intersects the circumference; and, opposite to the G and H points, we will find K and L, while opposite the points C, F, and A, we have the point N.

Now we must trace the GL and HK chords. The upper part will correspond to the summer, the lower one to the winter. Divide each of these chords into two equal parts, and mark the median points P and O, respectively. Now draw a line through these points and through the center A, cutting the circumference in points P and Q. This straight line will be perpendicular to the equinoctial ray: in mathematical terms, it is called the axis. Pointing the compasses on these two centers (P and O) then with their opening equal to the length of the chords (LG and KH), trace two semicircles, the one for the summer the other for the winter.

In the point of intersection between the parallel straight lines and the horizon line, we will have the letter S to the right and V to the left. From the H point, trace a parallel to the axis QP, up to meet the opposite semicircle in the G point, and from L an other parallel straight line to the left semicircle up to K. This parallel straight line GH is called a locothomus.

Now place the point of the compass at the intersection D of this straight line with the equinoctial ray; open the compass up to the point H of intersection of the circumference with the summer ray. From the D equinoctial center, trace the circle of the months, called the manaeus, with a compass opening equal to the distance DH of the summer ray. And so we will have the representation of the Analemma….

What Ptolemy and Vitruvius call Analemmas are representations of the sky in a particular condition: the sky is a sphere by convention projected on the plane of the local Meridian, in the moment in which the plane of the ecliptic is exactly perpendicular to it.

It has been asked if the Analemma of Vitruvius is or is not the same as Ptolemy's, since Vitruvius, as you can see above, does not clarify all the details that we can obtain from the Analemma, but he takes it as known by all. Therefore information is not available for appraising possible differences between the ideas of Vitruvius (or of the dialists of his entourage) and those of Ptolemy. We might well maintain that Vitruvius has little interest in Gnomonics, and that he limited his treatment to some bare notes, avoiding the details on purpose. But there are those who think that originally the treatment was more developed, and that the current state is the fault of the copyists who have rehandled the text.

It could be observed that Ptolemy does not deal with the Analemma for the purposes of gnomonics. He develops the matter of the celestial coordinates, to clarify the relationships between the possible quantities to measure in the sky, with the purpose of correctly specifying the positions of stars and planets. The fact that then the Analemma has proven useful for tracing solar clocks is an added benefit.

It is very probable that the treatise of Ptolemy was the scientific and critical development of an ancient Hellenistic theory, and that the "technician" of the epoch of Vitruvius used only some particularities of it in an uncritical way. This situation should not be a surprise: in geometry there are many theorems that exist independent of any practical utility (and there are many applications done mechanically, without thinking about their speculative origin). Some properties of the geometric figures are often neglected because they do not have useful purposes at the moment – only to be picked up again when we notice that they are not so useless after all.

Also it is a matter of what is “in” or “out”, if we want. In fact today the Analemma is almost unknown; at best, some aspects of it are in use without the users’ knowing their origin. And almost nobody remembers or refers to the Hellenistic origin of the theories on perspective or the theory of the astrolabe, that is of the "sphere in plane" (another chapter of the Almagest of Ptolemy).

The Analemma is a figure born of geometry (at least in the intentions of the first users) for astronomic purposes. But it can be variously "manipulated" for the purpose of obtaining all the useful data for the construction of dials.

To use it to such purpose we must use the star of planes with center O, considered like the center of the Earth or, in our case, like the vertex of the gnomon.

It is essential to choose suitably the conventional planes sectioning the celestial vault, in a way that facilitates the ordinate location of the hour points and lines.

We must remember that the "French" hour lines, in the conventional representation of the celestial vault, correspond to some great circles of the sphere obtained from the bundle of planes that have the polar straight line for axis: in other words, they correspond to a certain number of meridians of a globe. Therefore the bundle of planes constituted from the hour planes is notable for our purposes.

Also the lines of the italic hours correspond to arcs of great circles. The bundle that individualises it give origin to a cone whose angle at the vertex is equal to twice the latitude of the place. Two planes that differ by 12 hours share a straight line that is on the "equinoctial plane", that is on the plane perpendicular to the polar axis passing through O, which contains the equinoctial circle of the sphere.

An extremely important plane is that on which the Analemma is traced, containing the Poles, the Zenith, the Nadir, the line of the North / South direction: it is the plane of the Local Meridian, to which all the development that follows makes reference.

Generally, the following have importance for our purposes:

- a bundle of planes whose axes are the East / West line passing through the O vertex: planes always perpendicular to the Meridian plane.

- a bundle of planes whose axes are straight lines through O, parallel to the surface on which the dial is designed, and contained in the Meridian plane.

For dials designed on declining planes as regards the Meridian plane, the talk is similar, but it presents some complication. We will gradually see how to address it.

NOTE:

I do not pretend to do an exhaustive treatise of the matter; I want only to address some points. I consider it essential to address the more common cases in a reasonable way, and to give a sufficiently ample picture of the graphic possibilities offered by the model of representation of the sky, named Analemma.

The curious reader could start from these notes and go beyond, and discover how the Analemma is the hidden origin of many constructive techniques and many analytical relationships (above all in the trigonometric field) developed in the 17th century. Particularly, with the Analemma, it is possible to apply plane trigonometry in substitution of spherical, increasing the "average" accessibility of the problem.

CHAPTER I

In this first chapter, we analyse one of the "logical" methods to draw the useful data. In the following chapters variations will be considered, more or less useful, according to the "tastes" of the user.

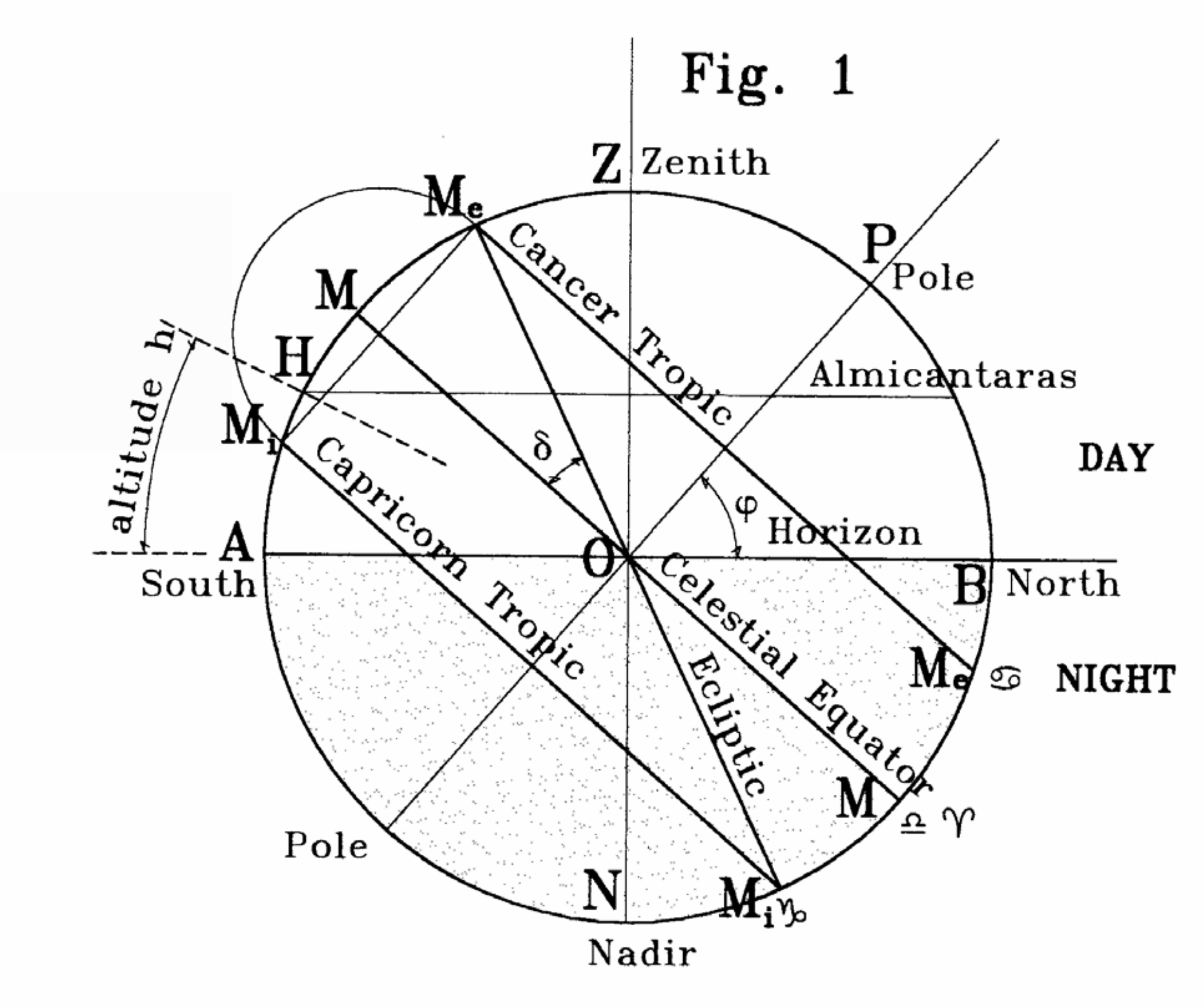


Figure 1

The Analemma in its simplest expression is the elementary representation of the projection of the meridian circle of a globe onto the plane, reduced to a few elements: the polar axle OP, tilted as the local latitude, the celestial equator OM, and the two tropic lines of Cancer and Capricorn (they are straight lines, but are the traces on the plane of the relevant circles). The ecliptic is reduced to a straight line (OM_e in the figure), the trace of the circle in the noon position of the Solstice day. Z is the Zenith of the observer set in O, and N the Nadir.

The point O has multiple functions, because it is both the center of the sphere - theoretically the point in which the fellow interested in what I will explain can be found, and the trace on the plane of the East/West diameter of the globe.[1]

Strictly speaking, the words “equator” and “tropic” are not quite correct, as related to the Sky, and not to the Earth, but we will use them here.

If you want to design the figure, note that the arc of the great circle (the declination of the Ecliptic) between equator and tropic is 23.5º.

The local horizon is a great circle whose trace AB passes through the center O.

The NZ line is the trace of the First Vertical. The First Vertical is the plane perpendicular to that of the local meridian and that contains the East / West line and the Zenith and Nadir points of the place.

The Earth is represented at the center O, reduced to one point. But it is not always so; at times it is convenient to consider the Earth like the whole AB plane of the local horizon.

The lines parallel to the horizon represent particular circles above the sphere, called Almicantarats or circles of altitude. (*If I rotate a telescope around a vertical axis, as for example with the theodolite, the telescope sees in the sky all the points forming the same angle - the same "altitude"- with respect to the local horizon. The line of sight traces ideal circles on the sky vault that we could call "parallels" of the local horizon, the Almicantarats*).

Figure 2 represents the Analemma still, and points to the search for the declination parallels, particularly those corresponding to the beginnings of the Zodiac Signs.

If we capsize the ecliptic (*i.e.* rotate it down into the plane), we can divide it in arcs of 30°, getting the points of passage between the Zodiac Signs. Obviously, on the projection of the ecliptic, the position of the Sun on any day of the year could be characterised (1° = 1 day, approximately and simplifying) and, at least in theory, the line of declination corresponding to any day of the year could be determined. (Here the pedants have a fine time, because a “fixed” angular value could not be attached to the date. It is known that the angle varies during the day; the Sun "moves" - tracing a kind of spiral. But we are satisfied with approximate values. After all, even the declination curves traced on the dials are only curves of first approximation, and they do not correspond to a precise indication of the apparent motion of the sun.)

Therefore we can project the series of points of passage on the trace of the ecliptic, and find their traces on the plane of the local meridian of the corresponding declination parallels. The figure that results is obviously conventional, but nothing prevents us from drawing only the declination parallel related to a certain day of the year.

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[1] Not quite: it should be the globe built in past centuries with the representation of the sky: constellations, meridians and celestial parallels, ecliptic and relative zodiac, *et cetera*. Then above all the astrologers used it, (if one could afford such an expense); but it was also an essential ornament of the "Studios" of the local squires, those with pretensions of humanistic affairs. A famous builder is Bleau, but magnificent examples are had from Apianus and from other builders, above all Dutch and German.

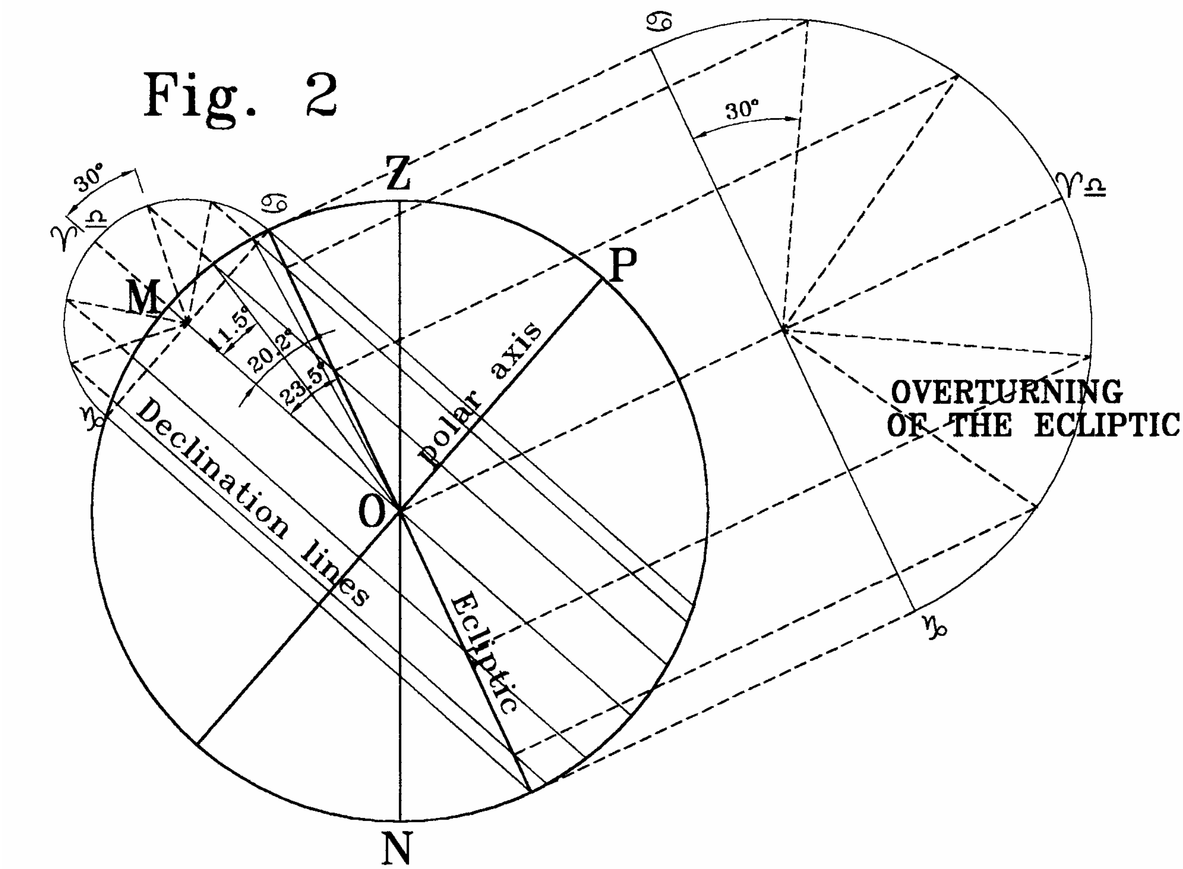


Figure 2

But the illustrated technique is not commonly used; it is preferred to trace the small semicircle on the equator, and to divide it in 6 parts (or in 180, if we want to refer to a date "almost" precisely). The result is the same. In the text of Vitruvius the small semicircle is called the MANAEUS.

In the figure we have not gone any further, but from this operation come the varied tools for tracing the curves of declination graphically (the names are varied, but they all correspond to the same tool: Sciaterre, Ray of Zodiac, Solar Radius, Sector of Declination) constructed in the past centuries.

Figure 3 refers to the use of the Analemma for the sketch of a horizontal sundial. Particularly here we propose the first step: the hour points on the Equinoctial line.

We suppose that the sun is on the equinoctial plane: if we capsize (with the usual technique) the equatorial plane, we could find the diurnal arc easily and the night-time one, and divide the diurnal arc into 12 hours. (In the sketch we did it on a half circle, tracing only the hour points of a half day).

The figure explains how it is possible draw the hour point above the dial, departing from the representation of the Analemma. Let OK be the vertical gnomon (and OC if you want it tilted).

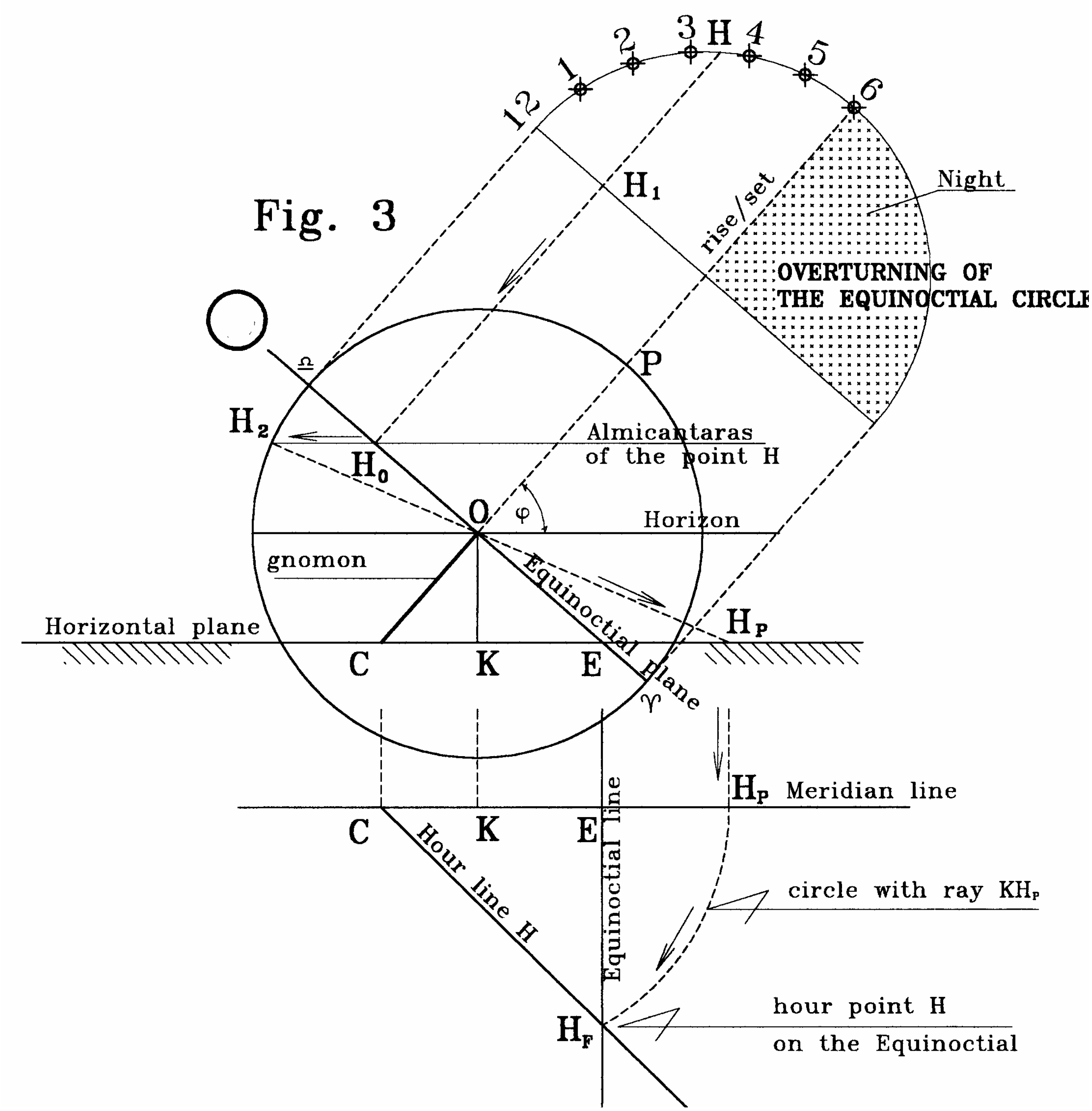


Figure 3

The point E is the trace of the intersection between the equinoctial plane and the dial plane. The dial is represented under the Analemma; CE is the meridian line, and the perpendicular through E is the equinoctial line.

Let H be the position of the sun to be represented by the projection above the quadrant. Project H in H_1 above the diameter, and then in H_0 on the equinoctial plane of the Analemma.

If we pass the relative Almicantarat through H_0, we get the H_2 point, which has the same altitude as the H_0 point and of the H point in which the Sun is found. We in practice could imagine having rotated the real position of the Sun up, to superpose on the meridian circle. We now could project its extreme H_2 in H_P.

Considering that all the points of the Almicantarat circle H_0H_2 can be projected on the horizontal plane forming a circle (because the plane of the Almicantarat is parallel to the plane of the dial), it will be enough to trace an arc of the circle with radius KH_P up to the equinoctial line on the dial, and thus get the H point, the shadow of the gnomon's endpoint.

It may be objected that this operation is a useless complication of the graphic technique commonly found in all the gnomonics texts, that consists in turning over the whole equatorial plane above the plane of the dial. It is true, but it could also be countered that the operation

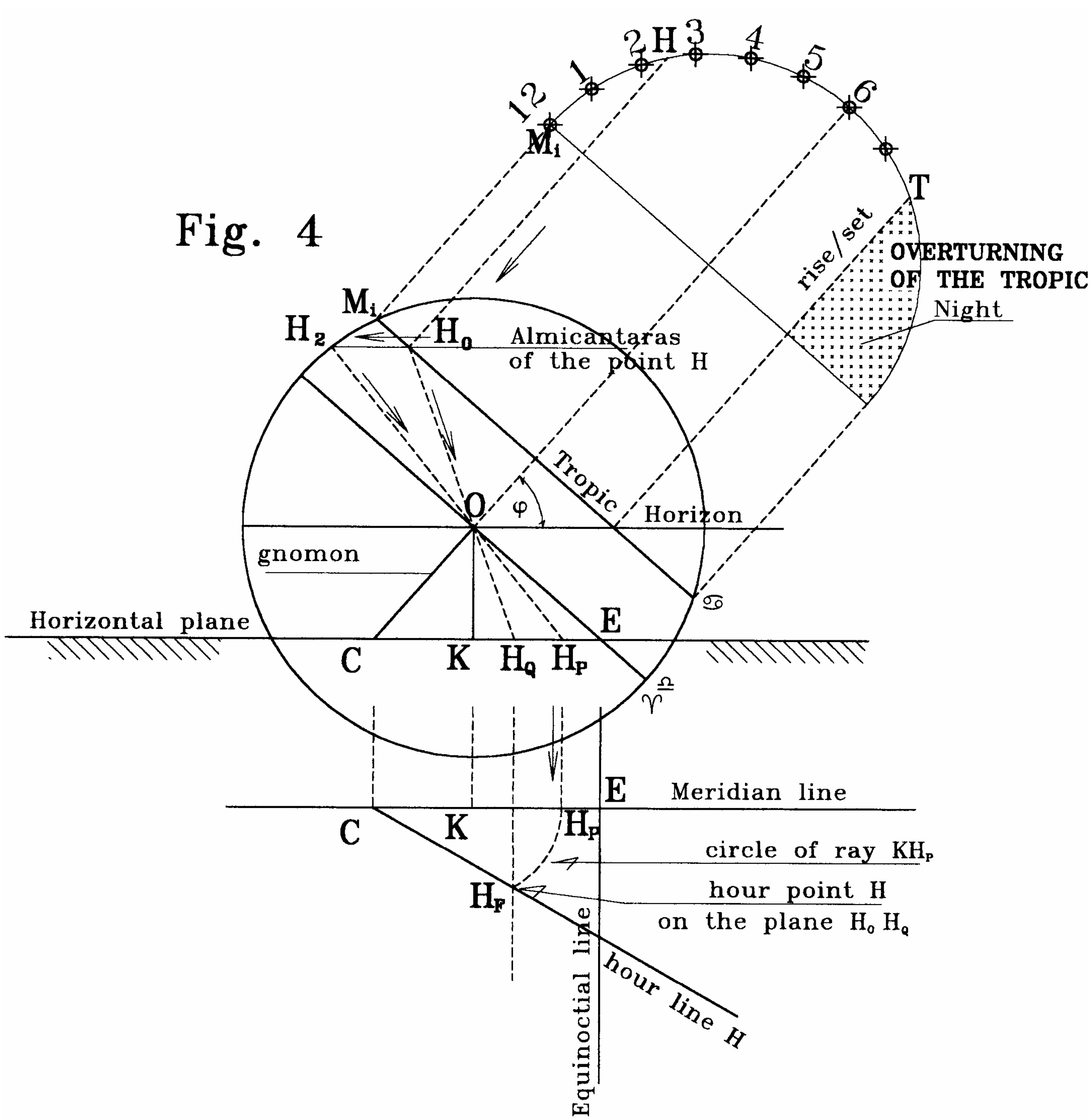


Figure 4

commonly used is but a by-product, a particular case of application of the Analemma, only useful to find the points above the equinoctial line.

Figure 4 refers still to the sketch of a horizontal sundial.

You must imagine now that the sun is in one of the declination parallels. The figure is traced for a point in Cancer because the hour lines are segments of a straight line, and it is preferred therefore to use the extreme circles, that allow us to find the extreme points of the hour lines. However, to represent the lines of the hours near to dawn and sunset, it will be necessary also to appeal to the parallels of intermediate declination, since the development of hours in winter is clearly briefer than in summer.

Capsizing the tropic, the resulting figure illustrates the M_iT length of the diurnal half-arc (and obviously of the night-time one). It is possible therefore to divide the half-arc into the number of hours that comprise it (If for example it is applied to the italic hours, it will be necessary to design the whole daily arc, because the hour points are not symmetrical as regards the noon point: the point of the 24 hour corresponds to the T point, and you must come back, 15 degrees per hour, along the circle).

To represent the projection of the H hour point, the method is very similar to the one illustrated above: you find H_0 on the diameter, and through this you pass the Almicantarat H_0H_2. These points are then projected: H_0 in H_Q and H_2 in H_P. H_0H_Q is the trace of a hypothetical plane perpendicular to the plane of the local meridian, a plane that contains H also.

On the horizontal sundial it will therefore be enough to pass a line from HQ (the other trace of the plane above) parallel to the equinoctial line, and to trace an arc of the circle with radius KHP, and get the HF hour point desired, the projection of H onto the declination parallel .

Obviously the operation must be repeated for all the hour points.

An axonometric vision of the layout has also been traced, that explains the mechanism of the crossed planes, to find the HF projection of H on the dial.

In substance, we have found two coordinates:
- the first, KH_Q, is in substance an ordinate along the meridian axis, beginning from the origin K;
- the second is the radius of the projection of the Almicantarat on the plane of the dial.

Figure 4bis shows a variation on the operation illustrated in Figure 4, and how you could undertake the operation compacting all into one sketch.

Here is represented a second method to find the position of the desired hour points: consider that the H_0H line of projection of the hour point H onto the diameter of the declination parallel is a horizontal line, above the relevant Almicantarat.

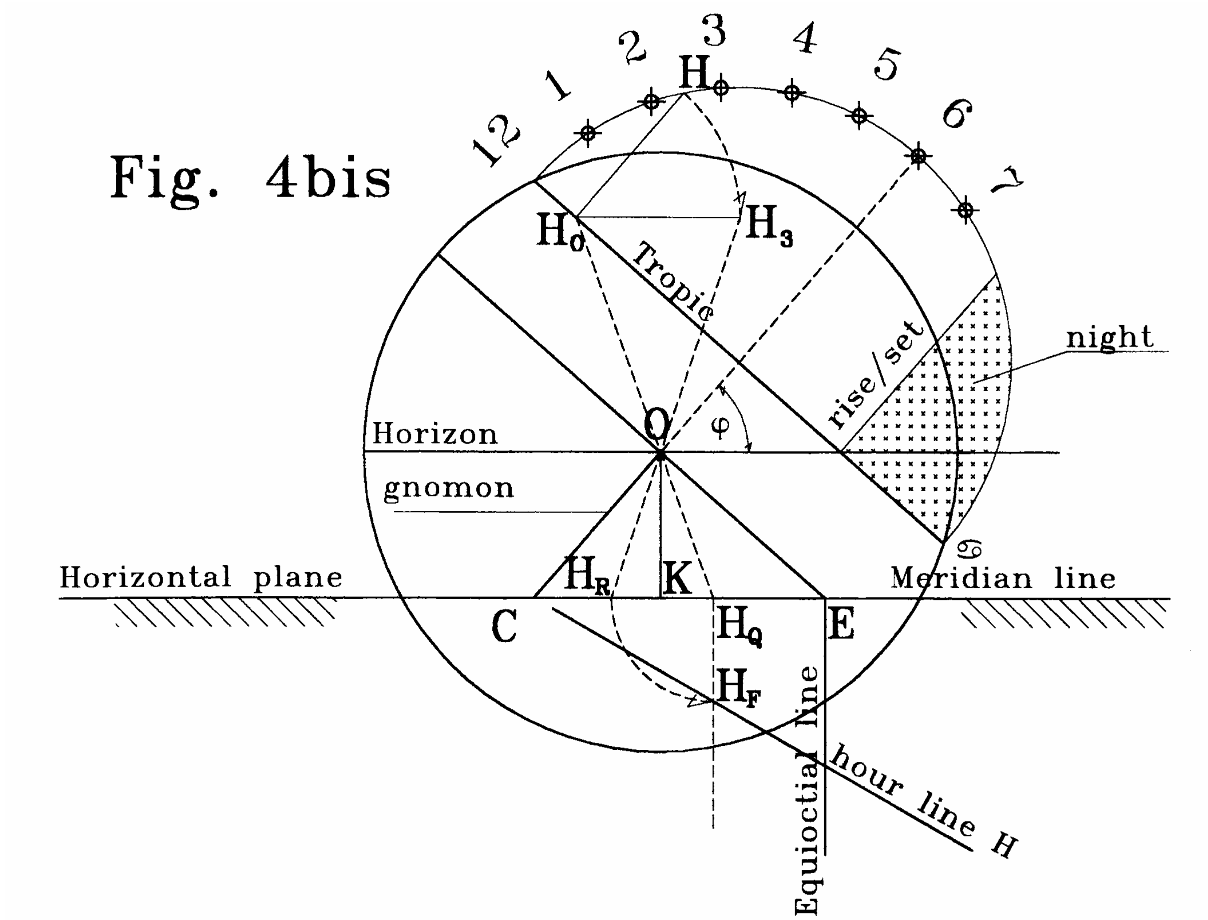


Figure 4bis

Rotate H_0H on H_0H_3, horizontal. The projection of H_3 will be H_R. (in other words, in the same sketch we did a new turnover, representing in a deformed way what happens on a plane perpendicular to the meridian plane - an operation justified by the theorems on similar triangles and on parallels cut by a transverse line.)

H_0H_Q is the trace of the mentioned plane, for which from H_Q the straight line parallel to the equinoctial is made to pass. On it, trace the H_QH_R distance.

In other words: these operations allow us to find the two co-ordinates of all hour points, with respect to a system of axes constituted by the Meridian line and by the trace of the First Vertical.

The two techniques (illustrated by figures 4 and 4bis) can be used together; one to control the quality of the other.

Also here we have two “Cartesian” coordinates on two perpendicular axes with origin K, one of which is the meridian line.

Figure 5 is "double," that is, in it the two techniques illustrated in Figures 4 and 4bis are

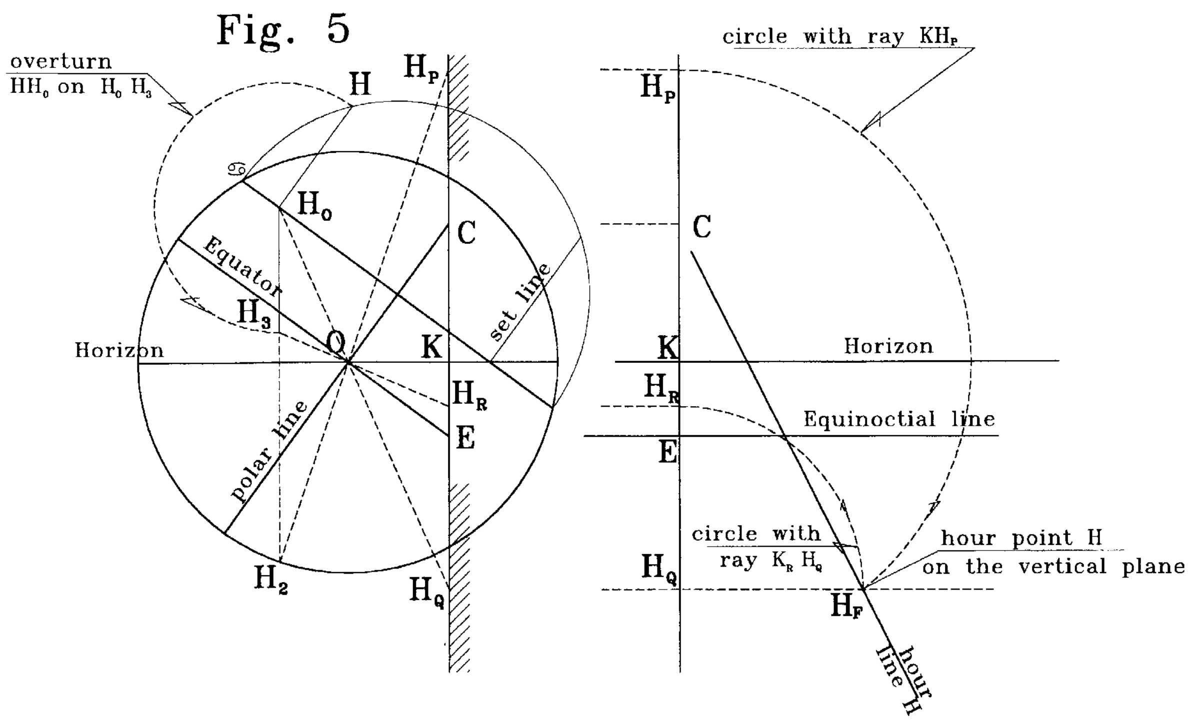


Figure 5

combined: the operation of the preceding figures is repeated for a vertical nondeclining sundial.

1) the projection of H_0 will be H_Q; the straight line H_0H_Q corresponds to the trace of the plane perpendicular to the Meridian one. Therefore one coordinate of the shadow point corresponding to the H position of the sun will be KH_Q.

To get the other coordinate, capsize H_0H on H_0H_3, on a vertical straight line (better: on a straight line parallel to the plane of the dial, that could not be vertical).

The proportional distance to H_0H_3 is H_QH_R. H_QH_R will be rotated on the horizontal straight line through H_Q therefore, finding the H point on the dial. (The operation is similar to the one illustrated in Fig. 4bis.)

2) Alternatively, with an operation analogous to that of Figure 4, we can also find the H_2 point (on a circle that is analogous to the Almicantarat, but feigns the sphere with a vertical plane) and project it in H_P. The H point on the quadrant is found by tracing the circle with radius KH_P.

Observe that the origin of the axes is always the foot K of the perpendicular stylus.

The figure 5bis illustrates another possibility, that perhaps is more general, or can be generalised.

The problem always consists in the search for a method to find the abscissa of the point of shadow (in the figure, the quantity H_QH_F).

On the sketch of the Analemma, consider the projection onto the horizontal plane of the Tropic

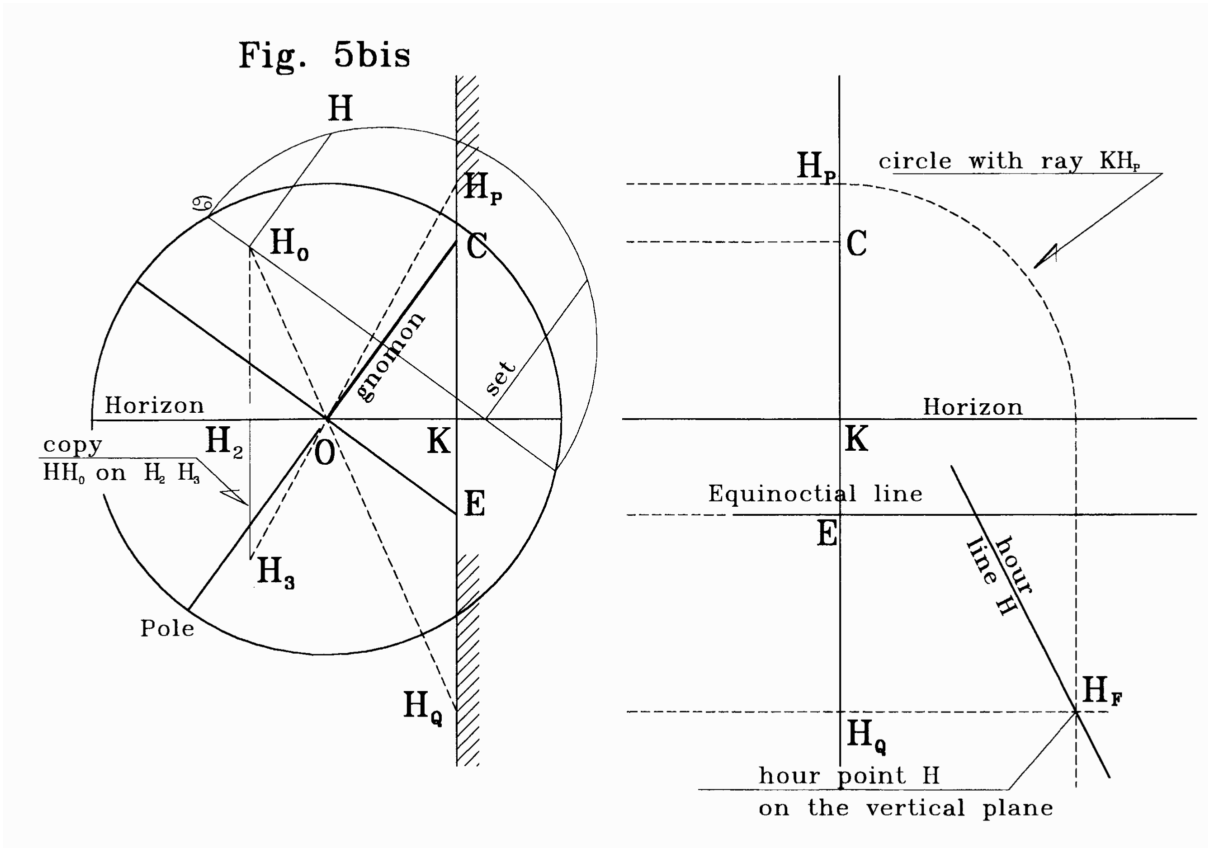


Figure 5bis

circle; it would be an ellipse. But there is no need to trace the ellipse: on a parallel to the wall, we can turn back on it, in H_2H_3, the distance HH_0. H_3 corresponds to the H point, viewed from the top, so the projection of H_3 is H_P. The H hour point on the dial is the cross point given from the two KH_Q and KH_P coordinates.

For vertical declining walls (Figure 6), the problem is more complex (but the solution corresponds in many points to that explained in Figure 5bis), because it requires us to imagine a plane that passes through the H and H_0 points on the parallel of declination and through the O point, and to find where this plane cuts the surface of the dial.

I repeat it: from the point of view of the geometric projections for a student of a scientific high school, one needs to build the ellipse, the projection on the horizontal plane of the declination parallel - in our case the Tropic. This was a construction that was not pleasant to our ancestors. We can accomplish the result more simply by bringing the distance HH_0 on the vertical line through H_0, to the side of the horizon line. It will be H_2H_3.

We need to draw the trace of the wall on the horizontal plane also. The H_S point is obtained, projection and trace, on the wall, of the vertical plane passing through O and H_3. (An analogous operation could be done for the symmetrical distance, finding a second H_S point, unless the projecting line through O does not ever meet the trace of the NK wall.)

Fig. 6

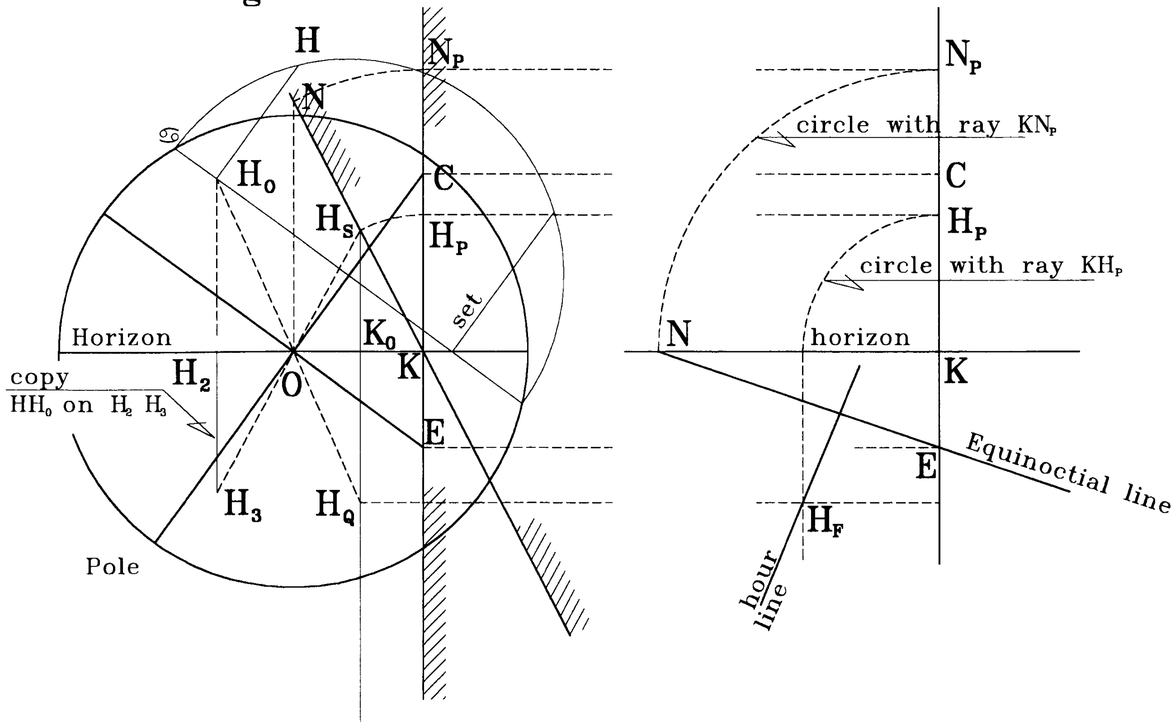


Figure 6

If from H_S we draw a vertical line (the trace of a nondeclining plane, passing through H_S), on it will be found the point H_Q, the projection of H_0. The dial is designed apart, finding the hour point H through the coordinates $KH_S = KH_P$, and K_0H_Q.

If a nondeclining plane is imagined (the First Vertical!) passing through O, it meets the trace of the plane of the dial. The KN distance is brought on the dial to find the equinoctial line, together with the KE distance, as illustrated in Figure 6.

In Figure 7, I want to illustrate a property of a particular construction regarding the Analemma: If from B and C, the extremes of the horizon line, the parallels BA and CD are traced, we get the representation of the regions of the sky that are "always visible" above AB, and "never visible" under CD. In one of the two parallels we therefore have the theoretical day of 24 hours, and in the other of zero hours.

Prolonging OA, the ED semicircle is traced now, with its center tangent at S. (*In fact it is not important that it is tangent, provided that its center is on SO, and the radius is SD, with D on line OA.)*

Given the parallel MQ, (here the Tropic is dealt with, but the reasoning is valid for any declination parallel) we can determine the length of the diurnal arc for that parallel by tracing OMR_0 and then R_0R.

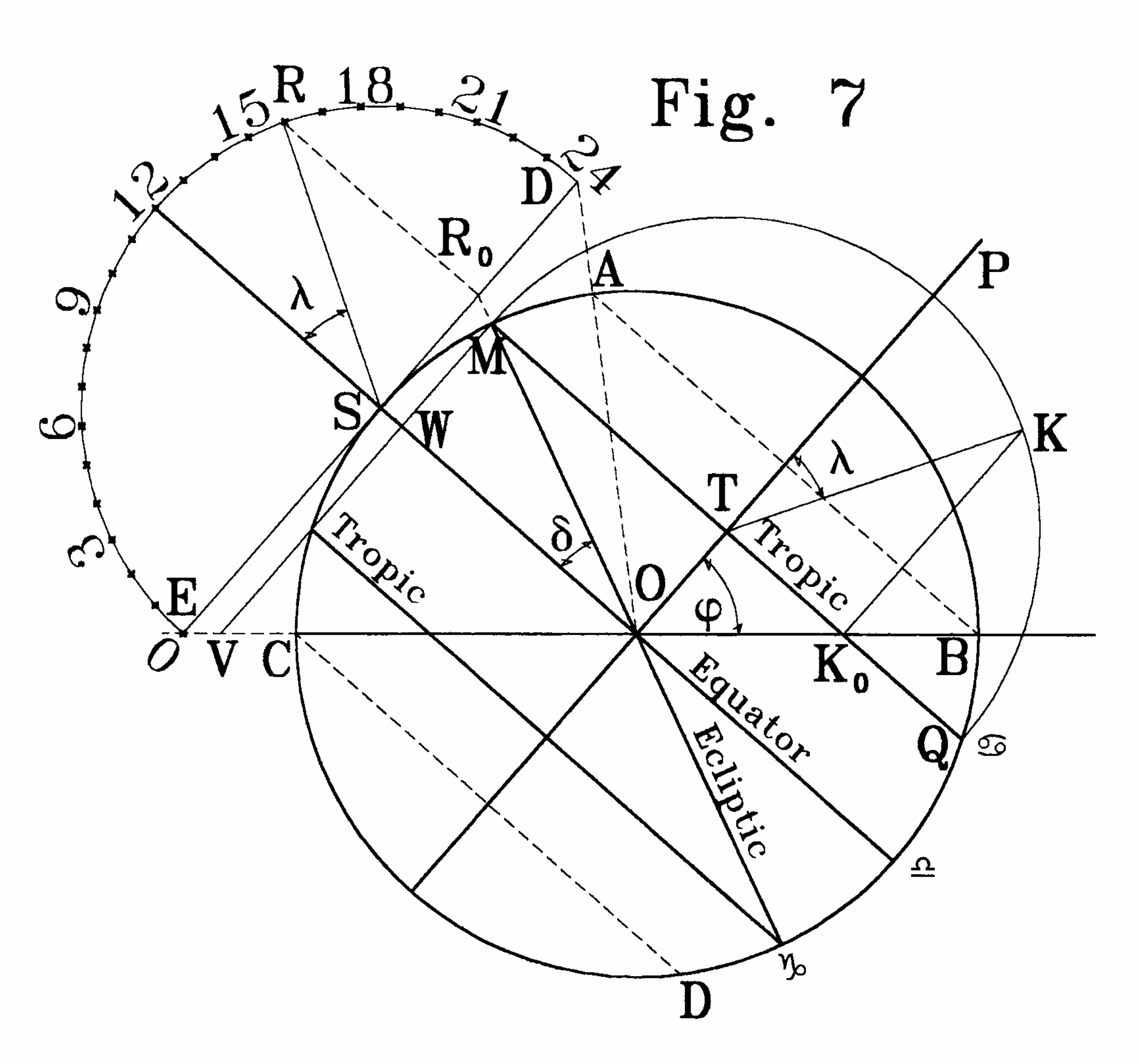


Figure 7

The ER arc, in relationship to the ED arc, is proportional to the length of the day. If we divide the ED semicircle into 24 parts, we can read the number of hours directly.

This operation is preferably used in the opposite direction, to find which declination parallel has a predetermined day length.

For example, Clavius suggests that we find the parallels that have day lengths of 6 hours and of 18 hours respectively. We can then find the "half hours" in each of the arcs and project them onto the plane of the clock. Uniting the "half hours" points of the arc of 6 hours with the "hour and a half" points of the other, the lines of the unequal hours are obtained, with tolerable inaccuracy, at least for our latitudes and for a wall dial.

The demonstration of the correctness of the method consists of finding a series of relationships between similar triangles:

$SR_0 / SR = SR_0 / SE = MW / WV = K_0O / OV = K_0T / TM = K_0T / TK$, from which is seen that the two angles designated λ are equal.

CHAPTER II

In this chapter we want to reconsider the matter from the beginning, in order to deal with the same problem with a method a little bit different. We must still find the extreme points of the hour lines, but perhaps with this new method the data are obtained in a better order, requiring less labor to transfer them onto the plane of the dial.

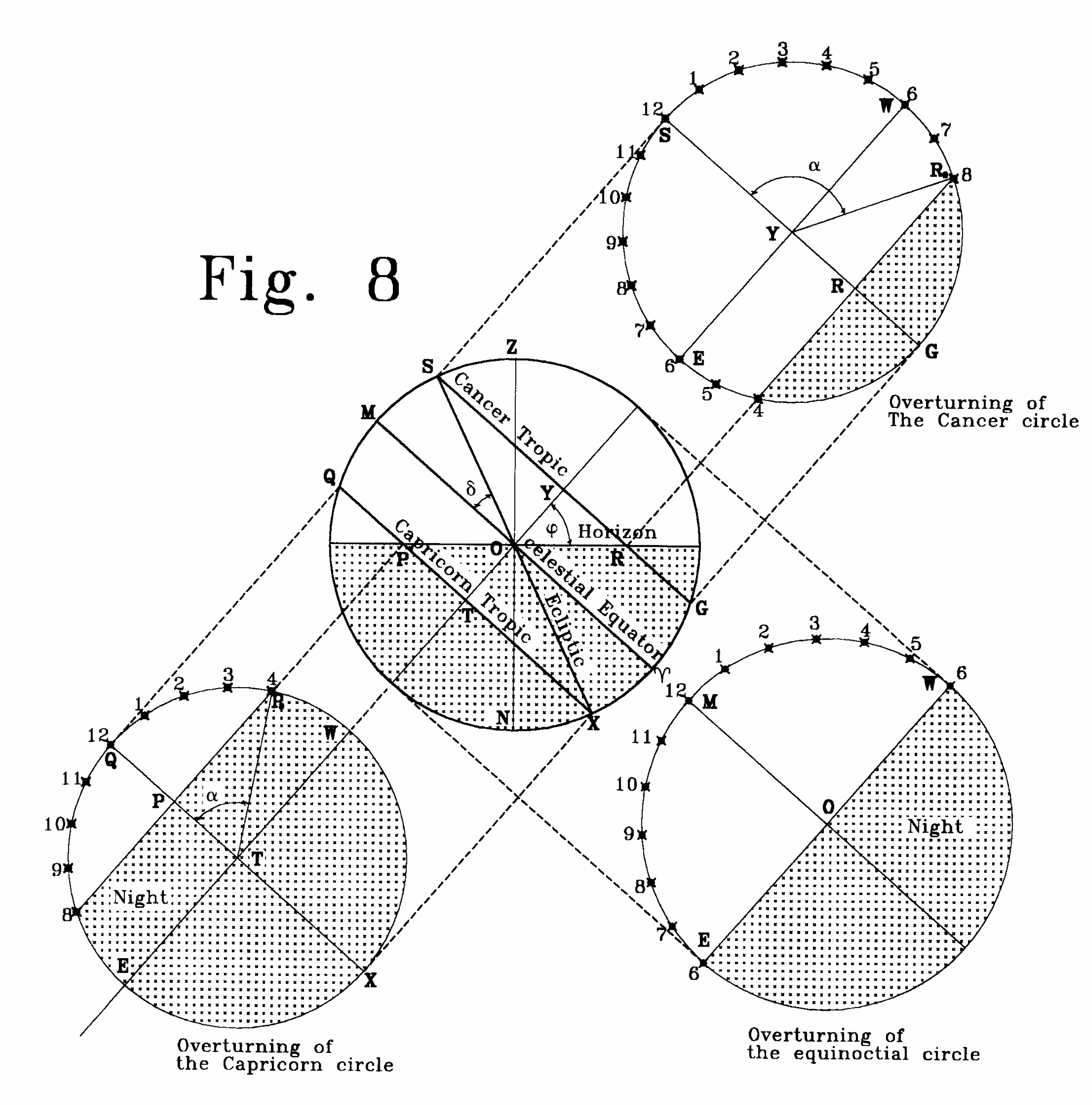


Figure 8

The method was illustrated by Muzio Oddi, in a treatise of 1614.

Figure 8 is substantially a repetition of one or more of the preceding figures, but it effectively illustrates the turnovers that allow us to find the diurnal and night-time arcs and to interpret the "global" meaning of the Analemma figure.

The extreme diurnal arcs α are found (relating to the OYR and YRR_0 triangles, for example) with the relationship: $\alpha = 90° \pm \arcsin(\text{tg}\,\delta\ \text{tg}\,\varphi)$.

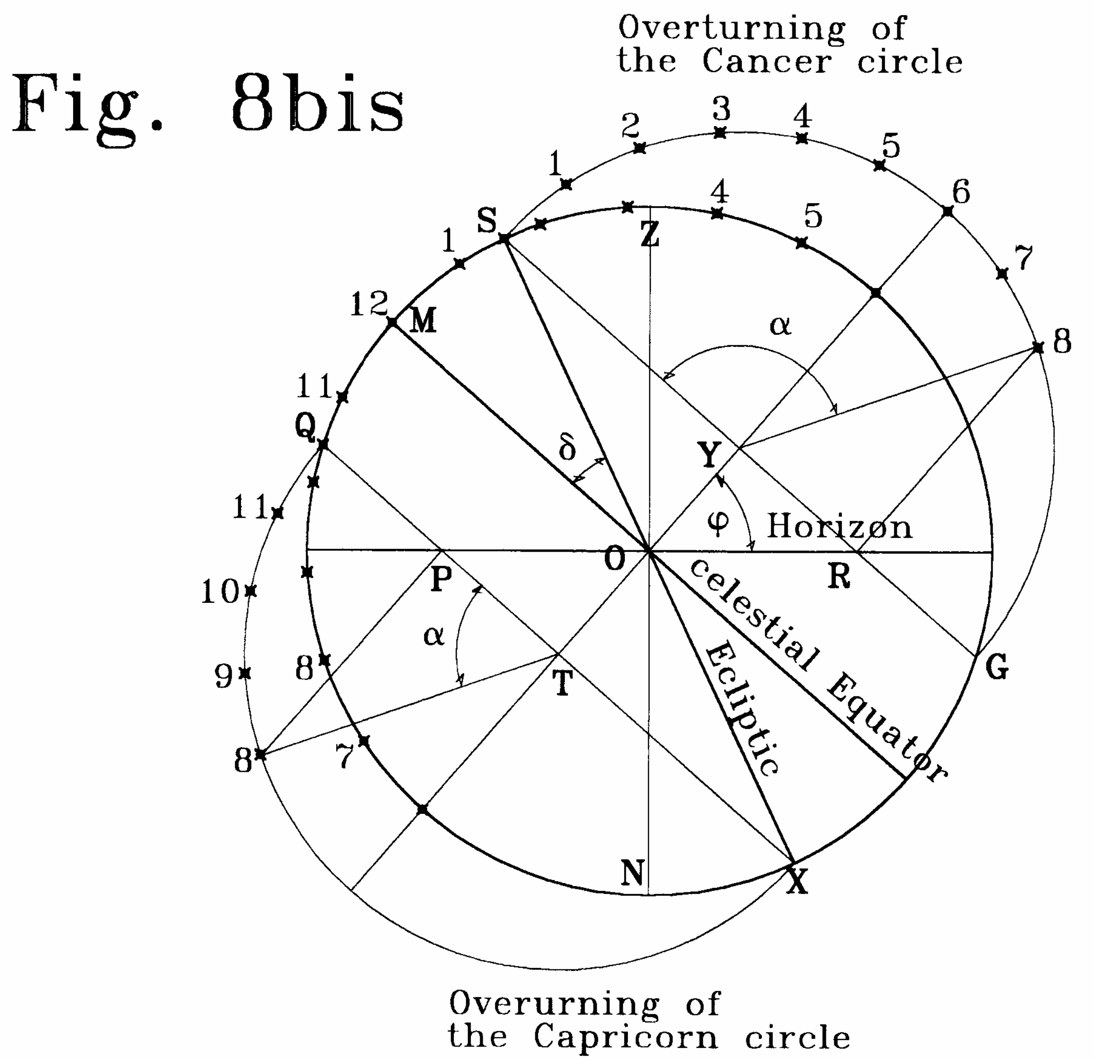


Figure 8bis

The representation of Figure 8 is usually "compressed" into Figure 8bis: the same meridian circle is also used for the turnover of the equatorial circle, which is superimposed exactly; the two semicircles, corresponding to the tropics are then traced, respectively with centers in Y and in T. The finding of the extreme semidiurnal arcs corresponds perfectly to the analogous operation done in Figure 8.

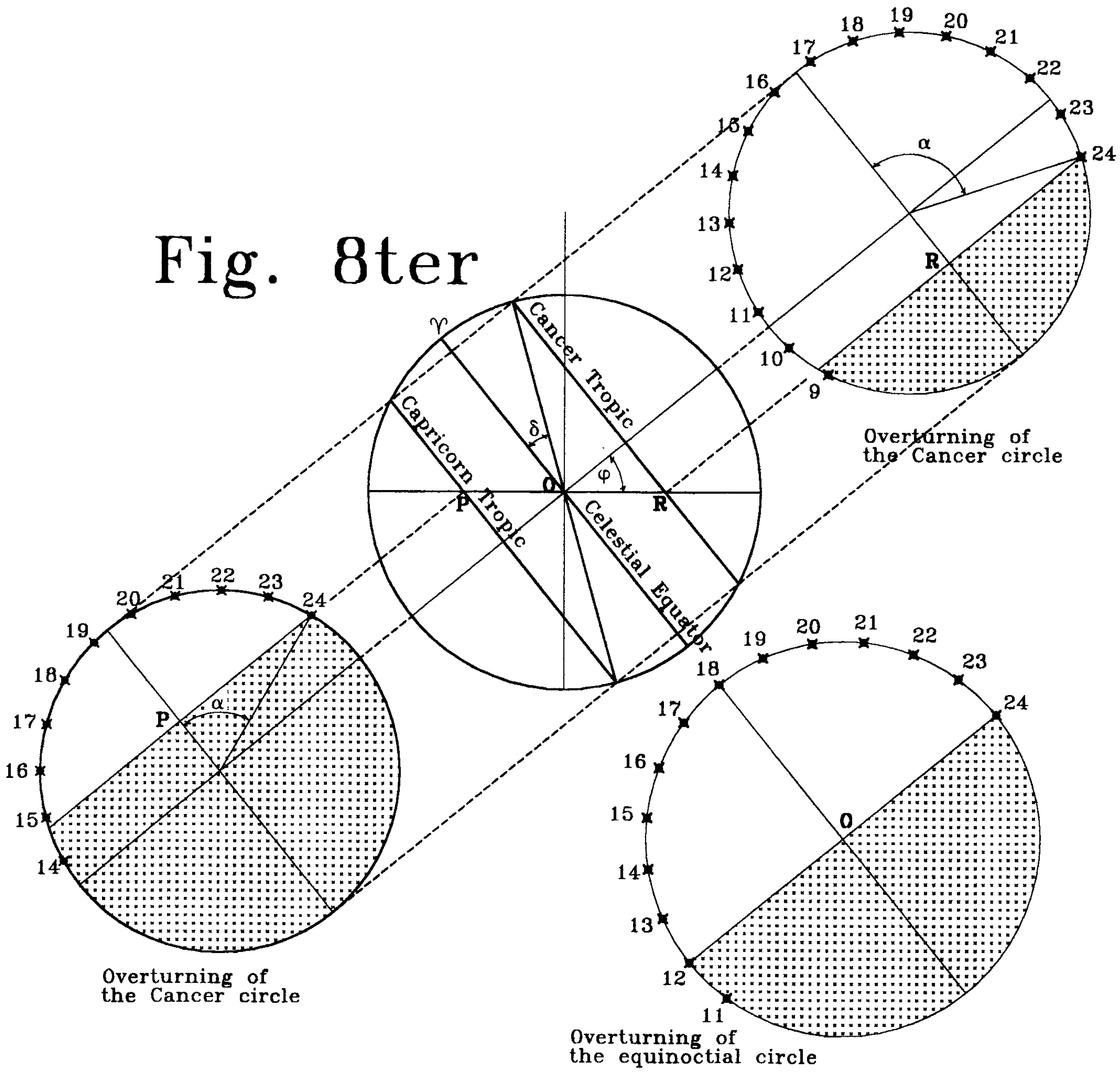


Figure 8ter

Figure 8ter shows the subdivision of the tropic circles according to the italic hours: it is enough to begin the subdivision into 15° arcs beginning from the sunset point. The figure will not be symmetrical as regards the noon line. The use of a figure like 8bis for the italic hours is inadvisable, because it is not possible to reduce the Cancer and Capricorn circles to semicircles; the superposition of the lines and of the hour points creates notable practical difficulties.

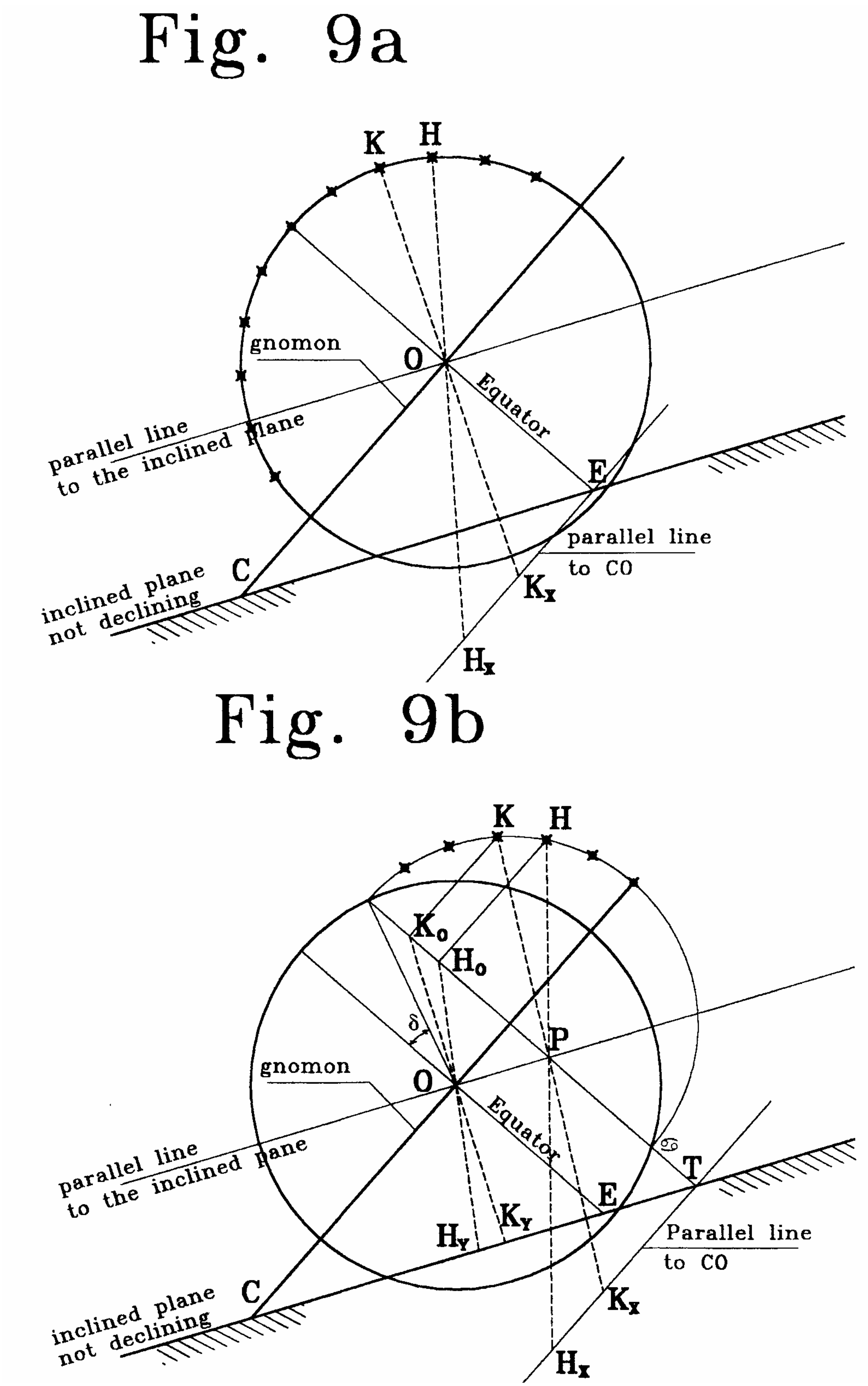


Figure 9a and Figure 9b

In Figure 9a, let CT be the trace of the plane (inclining but not declining, and therefore perpendicular to the Meridian plane) on which we want to trace a dial. OP is a straight line on the Meridian plane, passing through O, and parallel to the plane of CT. O is the vertex of the polar gnomon CO.

The equinoctial plane intersects CT along a straight line whose trace is E. It is perpendicular to the Meridian plane. All the hour points of the equinoctial circle project the vertex O of the gnomon onto this straight line. The construction of such points is illustrated in Figure 3, but it could be performed also as shown in Figure 9a.

You must imagine that the equinoctial circle and the straight line passing through E have capsized 90º around the equator line. The straight line through E is therefore a parallel to CO and will correspond to the equinoctial line of the dial.

From the figure, one can easily see the operation of projection on the equinoctial plane; the H_X and K_X points are the hour points of the H and K hours.

It suffices now to transfer the EH_X and EK_X distances onto the equinoctial line of the dial. The merit of this construction as opposed to that of Figure 3 is that the distances are "more orderly". We can project ALL the hour points onto the equinoctial.

It may be objected that this is in fact just the usual construction, and that there is nothing different from the better known methods. It is true, but this is also the premise to Figure 9b.

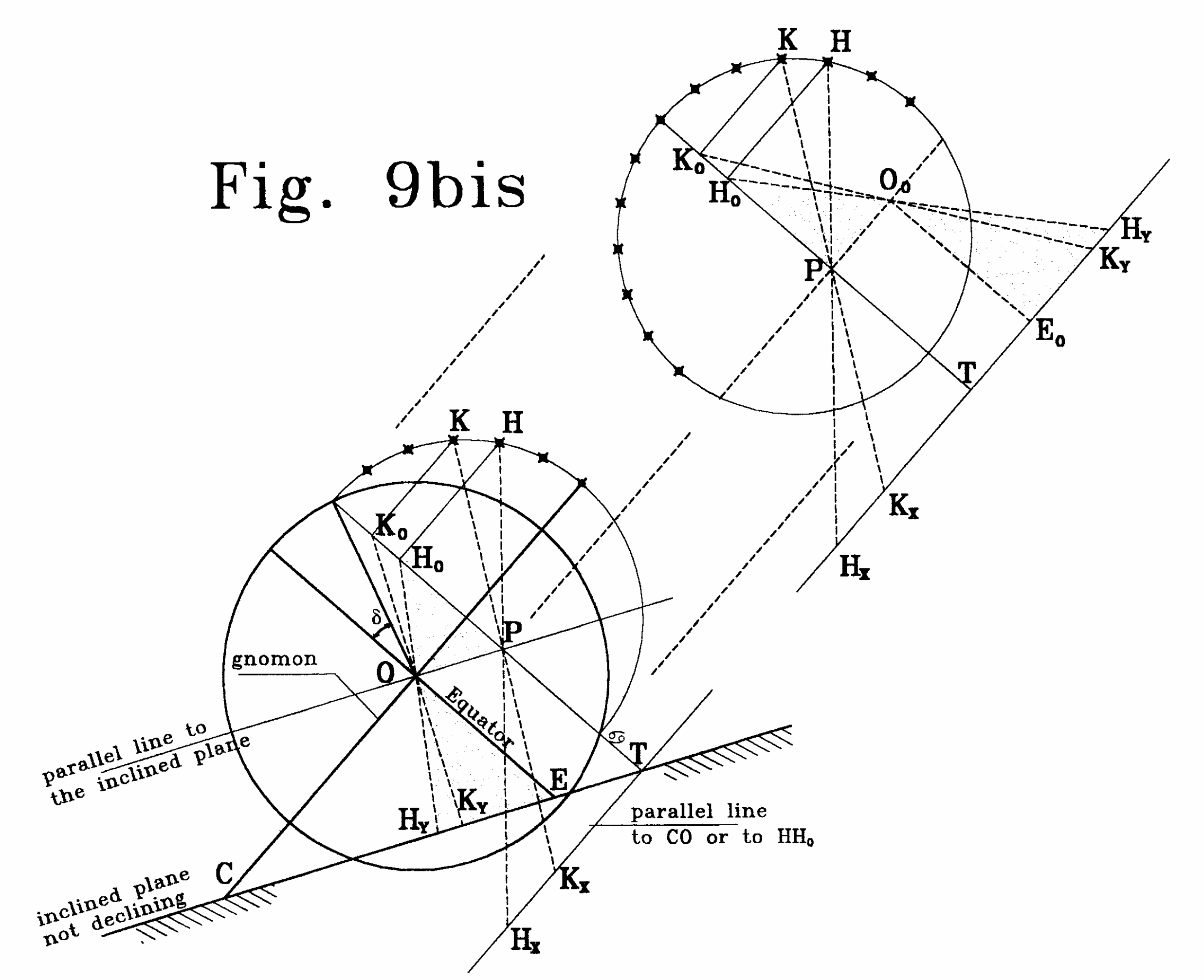
If we move on the tropic (but - we could say - on any declination circle of the Sun) the construction must allow us to find two coordinates: the abscissa will be taken on the straight line through T, and the ordinate on the Meridian line beginning from E. The operation is illustrated by the dashed-line triangles.

It is possible to reason in an analogous way as in Figure 9a: OE and PT are equal, since OP is parallel to the representation plane of the dial. And having traced TH_X parallel to CO, we conclude that the abscissas of the points H and K are TH_X and TK_X.

If the points H and K are projected in the points H_0 and K_0, and the projection of these last is performed through O, then the ordinates of the hour points of solstice are obtained as EH_Y and EK_Y on the CT line. (This operation is identical to what is done in Figure 4 with the straight line H_0H_Q.)

More correctly, it could be observed that a bundle of planes has been imagined, with axes coincident with the OP straight line: the planes passing through K, H, *etc.* determine a series of straight lines on the plane of the dial, parallel to the meridian line, and passing through KX, HX, *etc.*

Subsequently a second bundle of planes has been imagined, now with axes coincident with the East / West line passing through O. The planes that contain HH_0, KK_0 *etc.* determine straight lines on the dial, parallel to the Equinoctial line, passing through H_Y, K_Y, *etc.*



A decisive step: the Figure 9bis

So we have two series of ordinates, and we can determine the extreme of the hour lines directly.

Now a second step, decisive: In Figure 9bis, the declination circle is traced separately, with the PT distance and the perpendicular TH_X.

Trace also the straight line PP, always parallel to TH_X.

The distance PO_0 must be equal to OP, and could be traced to the right and to the left of P (here it is traced only on one side). The two couples of triangles underlined in the two figures have identical relationships between the corresponding sides. That is, the ordinate EH_Y of the lower figure is as long as the E_0H_Y of the higher figure.

The search for the X coordinate does not require comment, because what has been said above should be repeated.

Therefore in the same sketch, that to the right above, it is possible to get both the coordinates of the hour points, in an orderly, simple and rapid way: the "X" coordinates to transfer on the equinoctial line, are measured beginning from T; and the "Y" that will go on the meridian line, are measured beginning from E_0. Now the origin of the coordinate axes is the point of intersection between the Meridian line and the Equinoctial line.

What I call a "decisive step" consists in finding an efficient graph on which it is possible to trace all the coordinates without risk of confusion or of overlaps. And this sketch satisfies the demand.

E.g., in Figure 10 the horizontal Italic dial has been built.

The basic figure has remained decomposed into many parts to make the mechanism clear: From the Analemmas, the A figure has been derived, the equinoctial circle, turnover of the circle whose trace on the Analemmas is the OE straight line, and on this figure the hour points have been built with the method illustrated in Figure 8a. The hour points are symmetrical as regards the Meridian line, and therefore the dial is designed in only one part.

The figures B1 and B2 refer to the Tropic of Cancer: only one figure is dealt with, decomposed into two parts, to avoid graphic confusion. They could be added, but it is advisable to operate as illustrated.

An analogous operation is performed with the C figures, for the tropic of Capricorn.

The construction of the horizontal dial involves some observations of a geometric character, in addition to the data coming from the Analemmas:

The line 24 can not be traced, as the plane of the 24hour is parallel to the plane of the clock.

The line 12 is parallel to the equinoctial line.

The line 11 crosses the line of 23 in its equinoctial point, because the two hour planes are crossed on the equinoctial plane (as I noted above). Therefore, if a line of 10 exists, it crosses the line 22 in its equinoctial point.

Figure 10

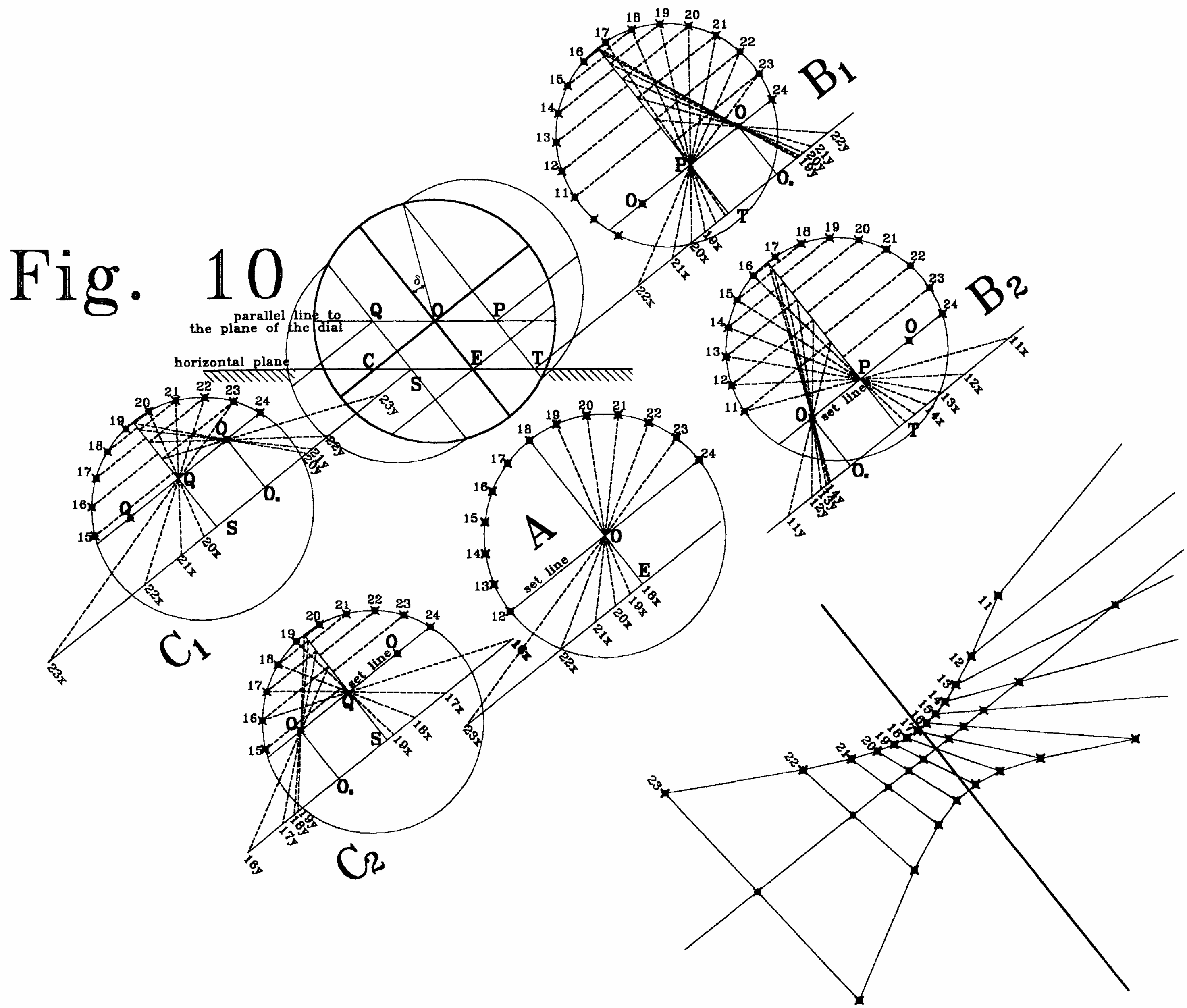
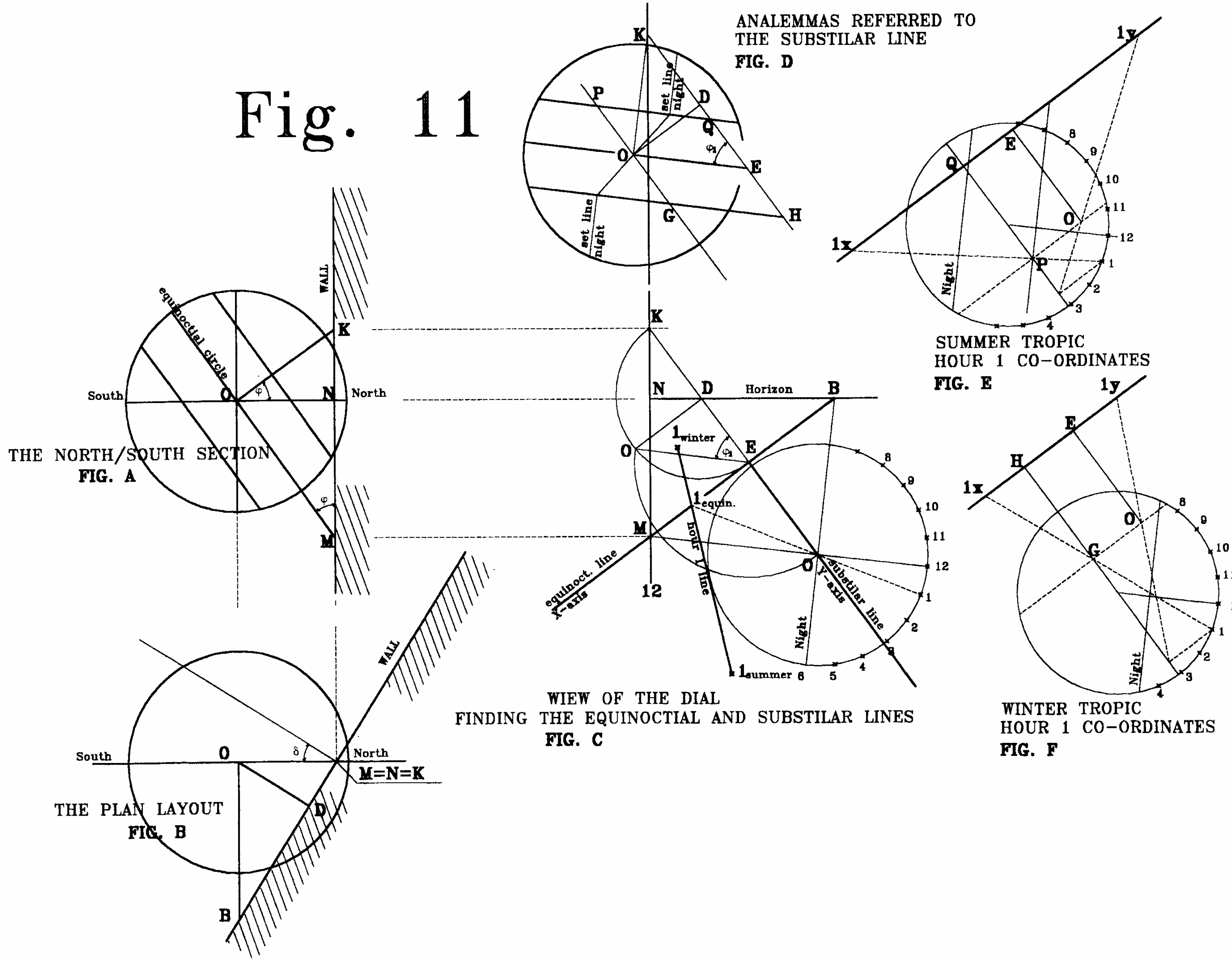


Fig. 10
parallel line to
the plane of the dial
horizontal plane
set line
A
B1
B2
C1
C2

Fig. 11


ANALEMMAS REFERRED TO
THE SUBSTILAR LINE
FIG. D
set line
night
set line
night
SUMMER TROPIC
HOUR 1 CO-ORDINATES
FIG. E
Night
WALL
equinoctial circle
South
North
THE NORTH/SOUTH SECTION
FIG. A
Horizon
1winter
1equin.
hour line
equinoct. line
X-axis
Y-axis
substilar line
Night
1summer
WIEW OF THE DIAL
FINDING THE EQUINOCTIAL AND SUBSTILAR LINES
FIG. C
WINTER TROPIC
HOUR 1 CO-ORDINATES
FIG. F
Night
South
North
M=N=K
WALL
THE PLAN LAYOUT
FIG. B

With Figure 11, I want to explicate the technique for building a vertical declining dial. It coincides with the search for the so-called equivalent dial.

The construction has remained decomposed into many parts for didactic purposes, but it is easy see how the figures A and B could be superimposed without creating problems of legibility; the same, but with results less legible, is true of the C and D figures.

From the two figs. A and B, the parameters for the construction of the dial Fig. C are drawn:

- NK and NM (where N is the horizon point on the Meridian line, K the center of the dial, and M the equinoctial point of the Meridian line) from figure A.

- NB and ND (where B is the intersection of the Equinoctial line and Horizon line, and the 6 equinoctial point, and D base point of the Stylus) from the B figure.

From the two figures, the length and position also of the perpendicular Stylus and of the polar stylus could be characterised, as well as the measurements needed to fix them on the wall.

The C figure, as noted, is the plane of the dial: on it, you can trace Meridian line, Horizon and Equinoctial line, and find the Substilar line. The construction as shown is known to any dialist. With the EK diameter semicircle we resolve OD (that you must set equal to the segment with the same name in figure B), the OE quantity, and the "equivalent" Latitude ω_1. We stress the importance of the OE distance, because it is the parameter of the following operations).

Also the construction of the right-angled MOB triangle is known to dialists: here we note that MO corresponds to the turnover of the position of the Sun at noon, instead that the substilar indicates the "meridian" plane of the Sun as regards the dial.

If we want to represent the Analemma of the substilar, we must project its equatorial plane on the surface of the dial, getting the circle with center O.

In our case we realise that the substilar meridian corresponds to about the 3 hour line. We will not consider the construction of the hour points on the circle, or of their projection on the Equinoctial line BM, because the construction is well known. We observe only that the position of such points is determined by the EO parameter.

We pass now to the construction of the local Analemma (D Figure). Its position above is only for didactic reasons: in it the line PG is parallel to the substilar KE.

I wanted to trace V and X on the tropic lines, correspondents to the analogous points of the Fig. A: they remind us, if necessary, that the "equivalent" Analemma is not independent from the latitude in which we are, and that the limits of dawn and sunset are those of the place, and not those of the equivalent latitude. Therefore the broken line UVXY has the function to remind us (but it could not be designed).

From these Analemmas, we derive the figures E and F; in the first, related to the summer tropic, we find the coordinates of the extreme hour points on the summer solstice, and in the second those on the winter solstice. In our case there we have represented only the construction for the hour point 1. Now the X and Y axes are respectively the equinoctial line and the substilar line, and the origin is E.

As further examples, consider two dials with unequal hours on vertical nondeclining walls in Figures 12 and 12bis.

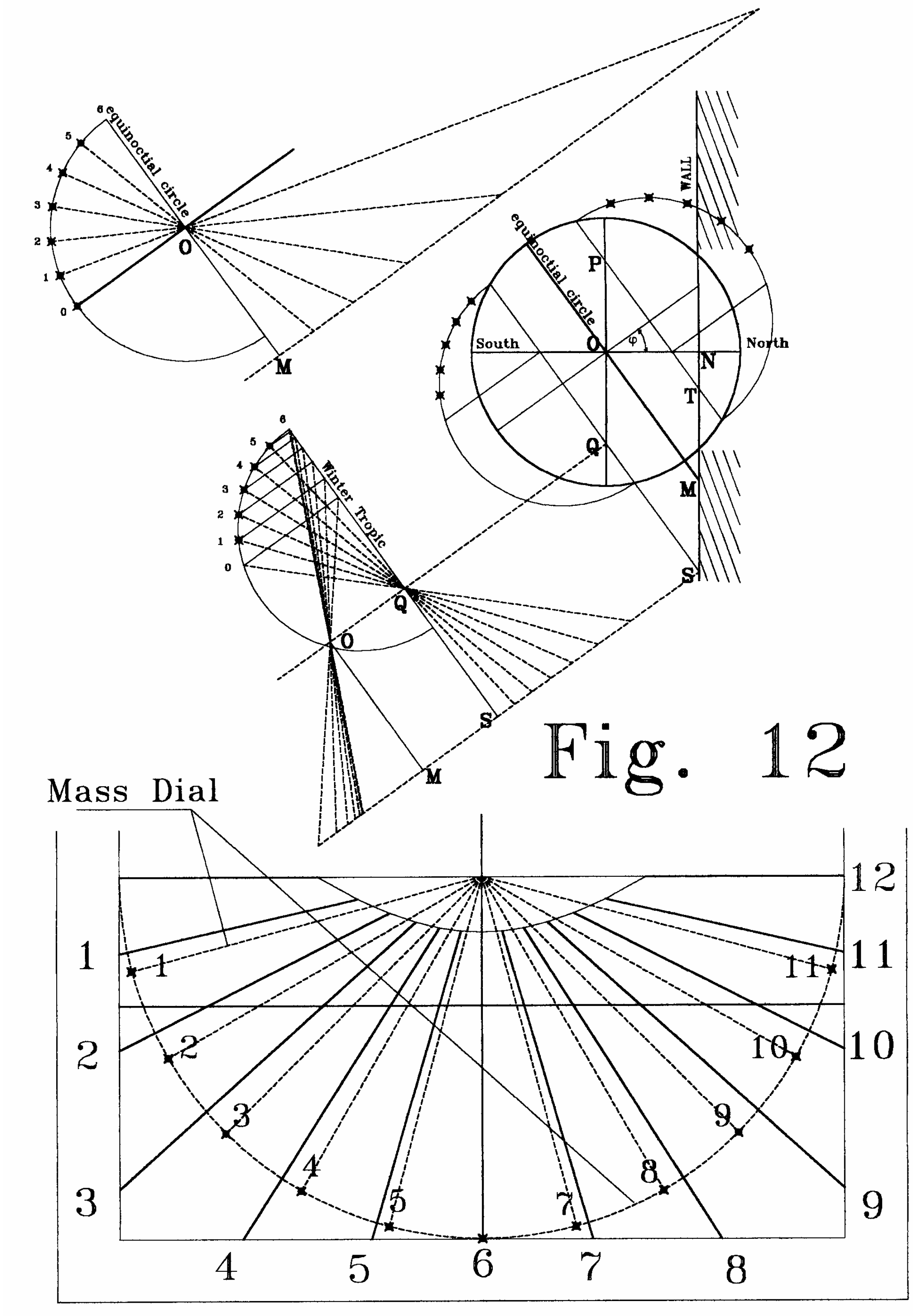
That of Figure 12 is built for 38º latitude. Observe that the graphic calculus for the summer tropic is not done, because the distances would come too great. It has been limited to only 4 central hours of the day.

Therefore the hour lines are built by uniting the points of the winter solstice with those of the equinox. The hour lines are considered straight lines, which is not true; if the reader has the patience to develop the Analemmas for a certain number of declinations (those of the Zodiac signs of Figure 2 could be enough), he or she could see the S shape of the hour lines.

The sketch of the dial may be compared with the so-called mass dial, typical of a certain historical period.

Figure 12bis is the development of the graphic calculus for a dial at latitude 51°: also in this case the comparison with the mass dial has been done.

The graphic calculus is perfectly identical in the two cases.



UNEQUAL HOURS DIAL ON A NON DECLINIG WALL

LATITUDE 38°

COMPARISON WITH THE MASS DIAL

Figure 12

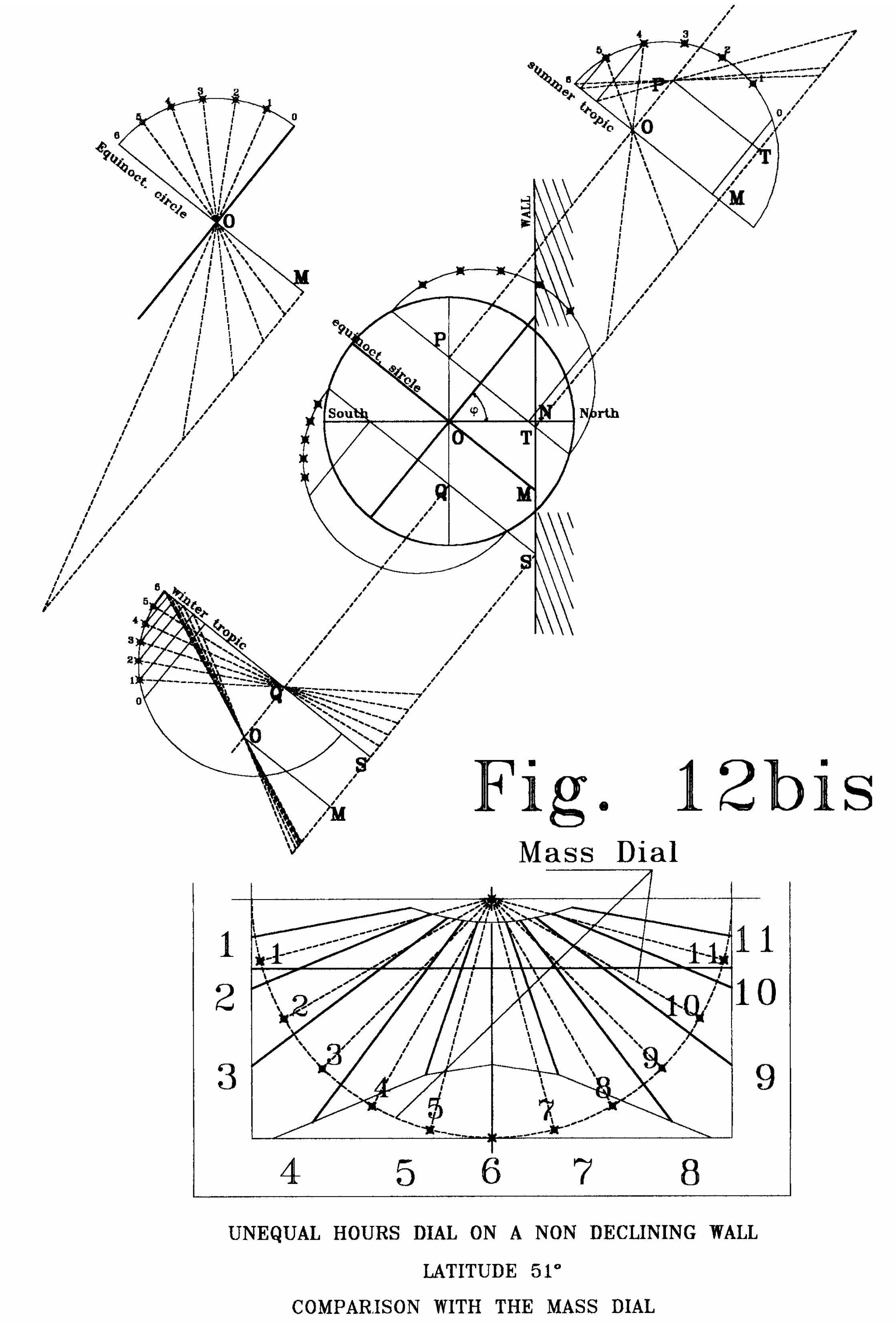


Figure 12bis

CHAPTER III

The Analemma lends itself to the construction of altitude sundials, since the finding of the Sun's Almicantarat in any position is essential to the use of the Analemma. *E.g.*, here are two graphics typical of altitude gnomonics.

The shepherd dial. In Figure 13 the positions of the sun are developed in 7 Analemmas at the ends of the zodiac signs. Naturally it is possible to multiply the lines of declination and to find the hour curves with more points. We will not dwell on the explanations, the figure needs no additional comment.

The dial that may require some explication is the so-called "tortoise". In substance it is the same as the shepherd dial, drawn (in our case) on 30^0 arcs: resulting in a figure whose form is vaguely reminiscent of a tortoise.

The figure of the tortoise was first mentioned, I think, in a table by Kircher, in his "Ars magna lucis et umbre," but it is not original to him. Kircher created the tortoise from a dial explained by Giovanni Paolo Galluci da Salò in a 1594 book (Nova fabricandi horaria mobilia...).

Galluci, less fanciful, enrolled it in a semicircle, and had built it (with 13 Analemmas) for the italic quadrant. Figures 13 (testudo) and 13b here illustrate the proposal of Galluci, and the use of the dial (the semicircle is rotated up to make the plumb line fall across the date, and the dial must be opposed to the Sun, so that the shadow falls along the thread).

In Figure 14 we show a quadrant, developing the graphic calculus of the 7 Analemmas of the preceding figure (Fig. 13). For illustrative purposes, only three analemmas are traced, that of the equinoxes, and the extreme two for the solstices. The construction of this kind of dial depends

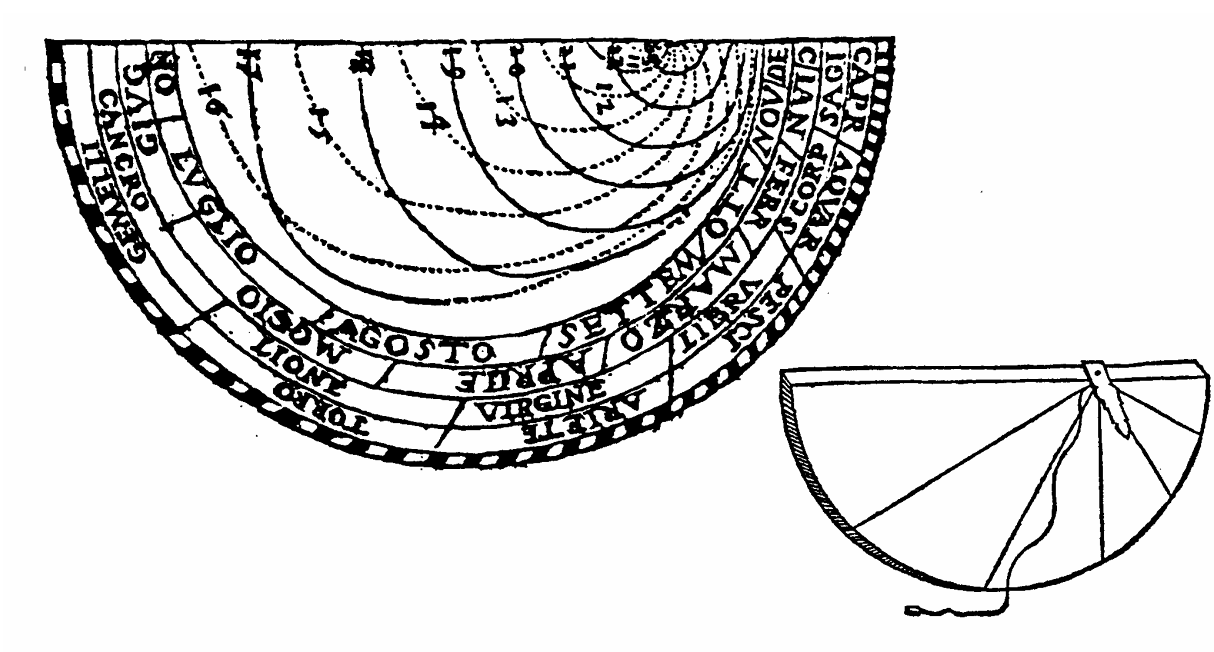


Figure 13b

Figure 13

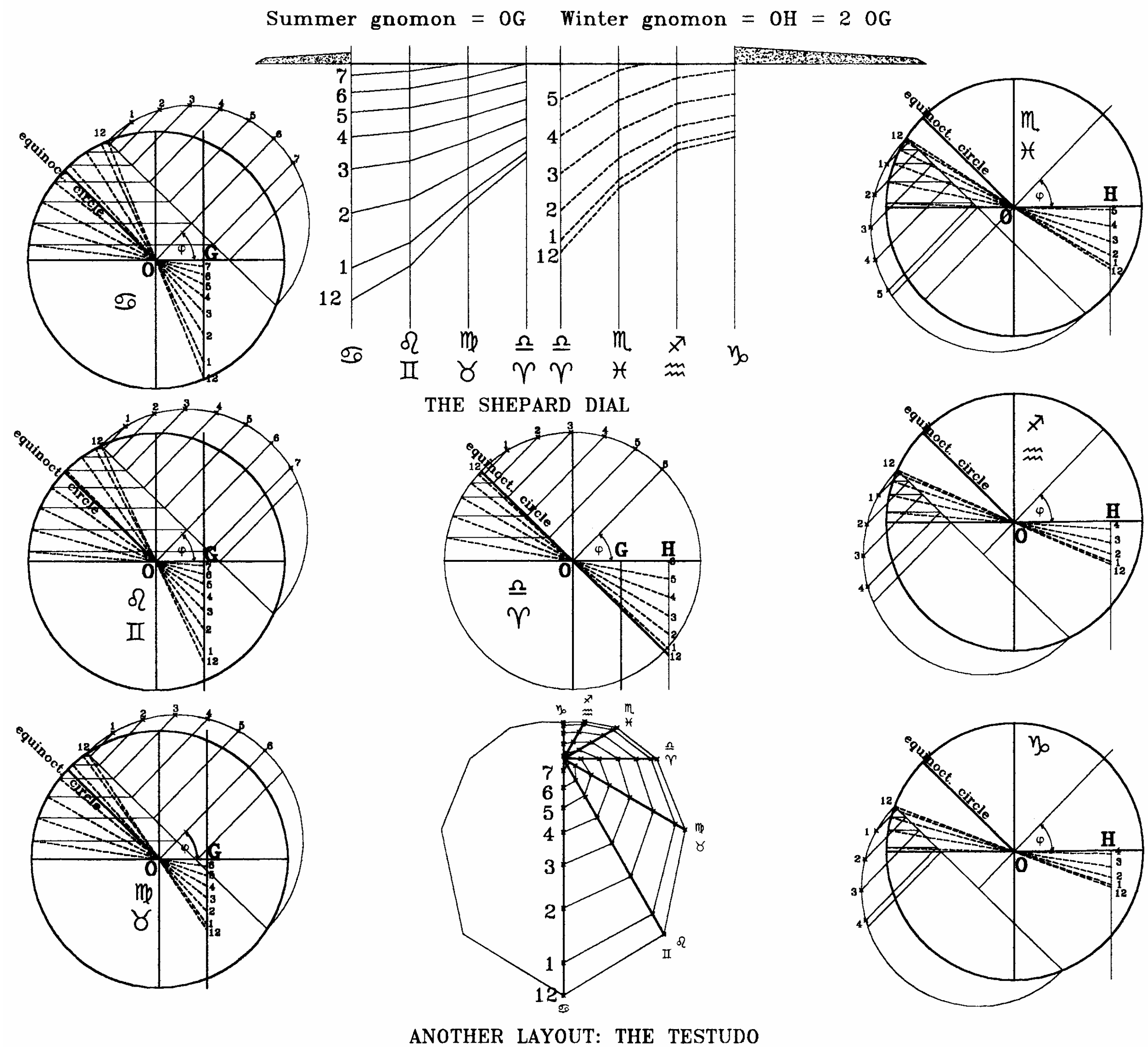


Fig. 13

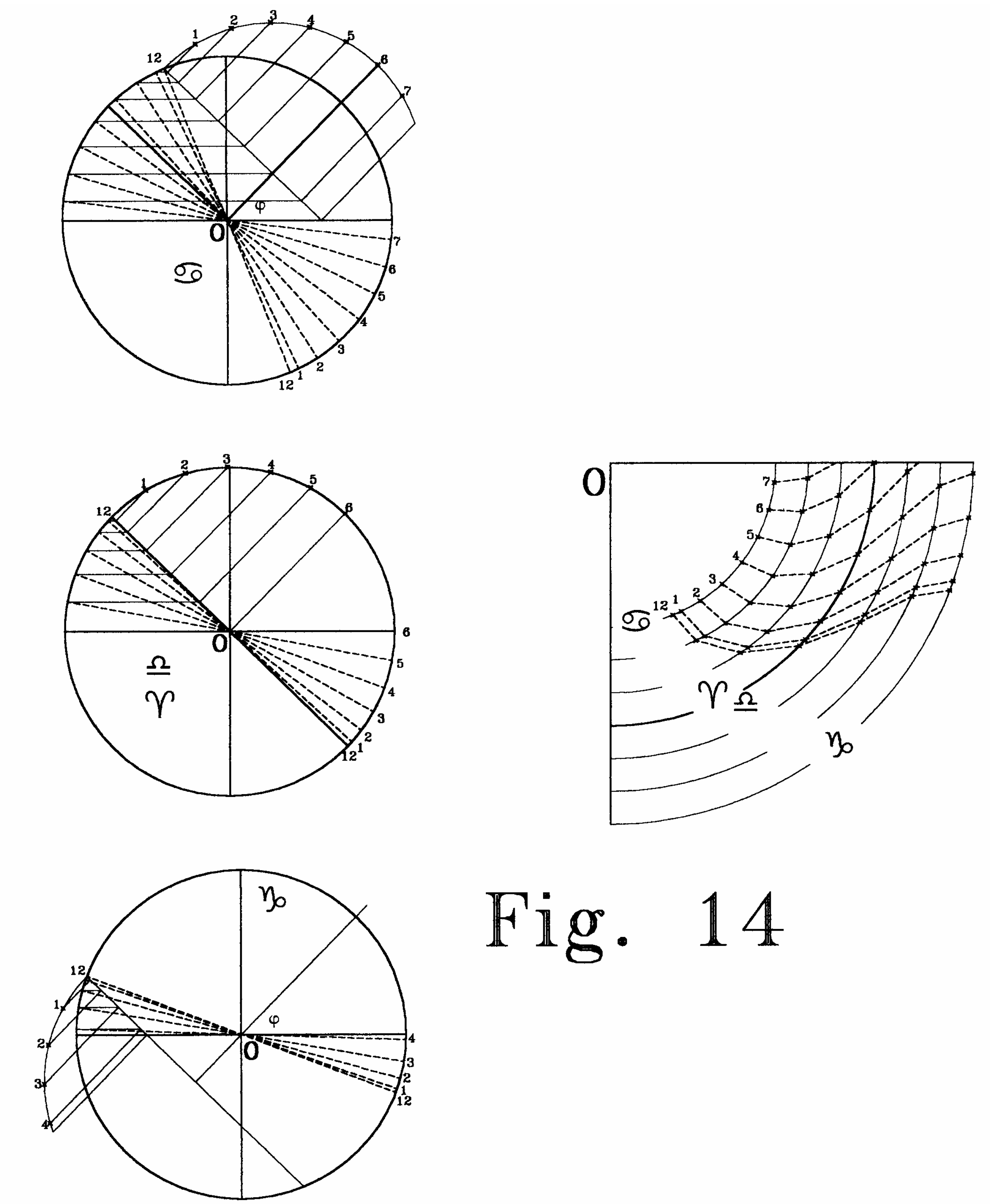


Figure 14

usually on the reference adopted for the determination of the horizon. If a plumb line is used the quadrant must be rotated 90^0. Also the distance between the lines of the dates could be relevant in the graphic presentation. Here I adopted equal distances, but it is possible to play with the intervals so that the hour curves become arcs of circle, or straight lines.

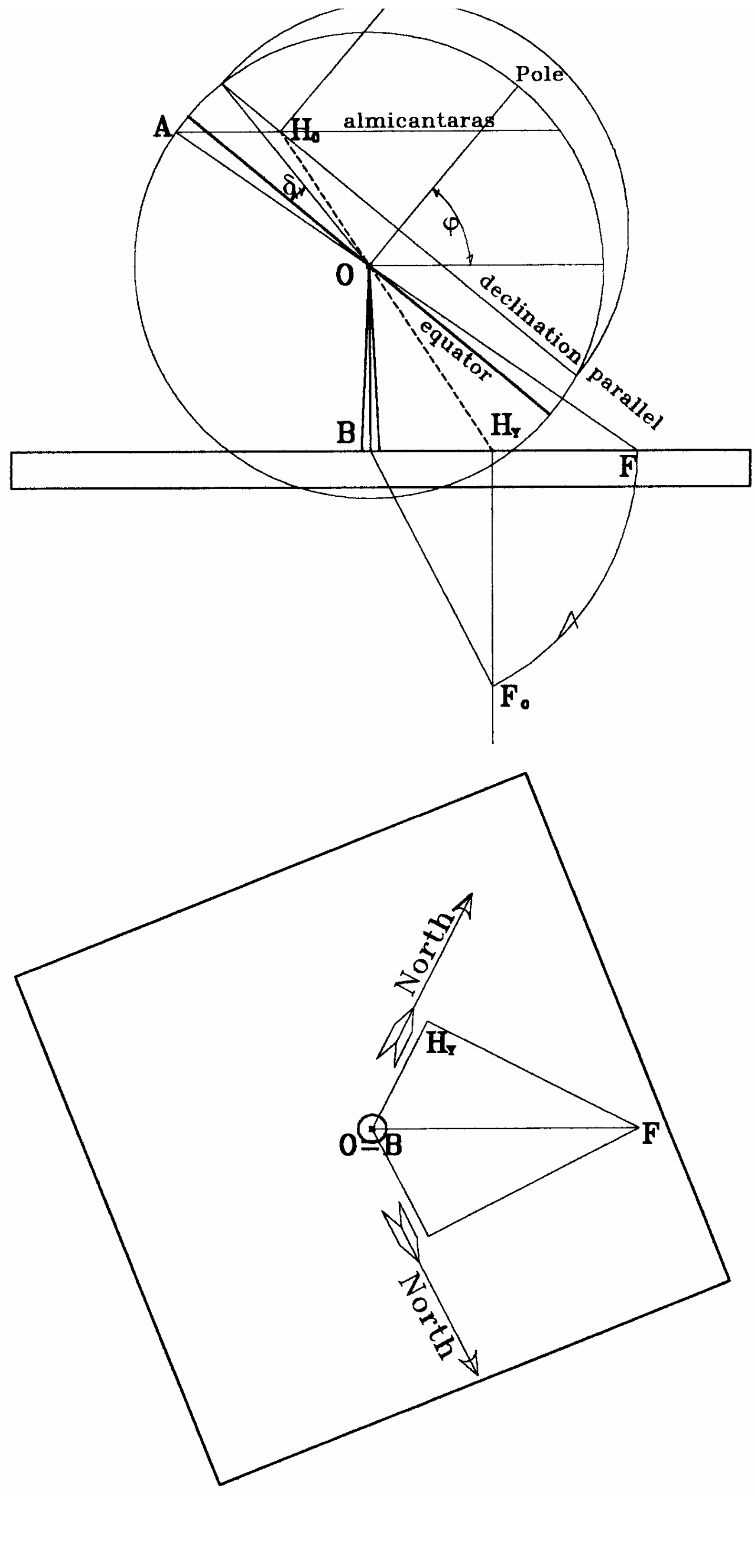

Pole
A
H_0
almicantaras
δ
φ
O
declination parallel
equator
B
H_γ
F
F_0
North
H_γ
O=B
F
North

Figure 15

The search for the Azimuth of the Sun

Let us solve the following problem: On a horizontal tablet, note the length BF of the shadow of a vertical stylus BO at a certain moment of the day; from this information, find the direction of the Meridian line. The day-date and the latitude of the place are known.

In Figure 15, let BF be the length of the shadow. I sketch in section the Analemma with center in O, latitude φ, and on it I find the declination parallel (of the day, that should be known from our knowledge of the date). Turn back the BF distance, the length of the shadow.

The way to follow is essentially reversing the procedure of Figure 4. The straight line FOA determines the point A belonging to the Sun's Almicantarat AH_0 at the given time (the Sun is therefore in H). The only thing that we do not know, in this case, is if the shadow point is referred to the morning or to the afternoon, that is, if H is to the right or to left of H_0. (But whoever took the shadow-point knows.)

Following the operations of Figure 4, we could project H_0 in H_Y, and determine the right-angled triangle BH_YF_0.

If we now transfer the triangle onto the tablet, making BF_0 fit with BF, we get two possible directions of North, coincident with the side BH_Y of the right triangle.

The problem is theoretical; I do not believe that anybody has ever arranged to find the orientation with this method. But it offers a further proof of the versatility of the Analemma "tool".

An Extremely Simple Graphic Method To Find The Parameters Of The Equivalent Sundial (applied to a declining and inclined dial)

Anyone who read the Internet mail list "Gnomonica" in 2001 would have been faced with a certain frequency of ponderous messages, relative to the so-called equivalent dial – all involving a flood of formulas!

And here am I, who claims to do all without any formula - the last beach; *après moi le déluge*. However I do it with a CAD program, as the results are to the level of the formulas, otherwise there is no reason to adopt this approach.

Let us start by clarifying an idea at least, at the cost of repeating what has been said, fixed and written by others from the 16th century on.

What exactly is the meaning of "equivalent sundial"? I would say that the explanation is more elementary if it is restricted to the most common type of dial that a dialist traces: the vertical declining one. If declination and latitude are known, the equinoctial line, the substilar line, and the angle between gnomon and substilar are first traced. Only afterwards do we trace the hour lines.

In other words, the dialist works on a nondeclining dial, that has the latitude of the complement of the substilar angle; the only difference is that when he traces the hour lines he does not write 12 under the substilar (even though it is the meridian line of the equivalent place), but under another line, that (by chance!) corresponds to the meridian line of the place where we are: the hour difference between our noon and the noon of the equivalent dial (I repeat: the substilar line) is the difference in longitude between our place and the equivalent one, and the substilar angle is the complement of the equivalent latitude.

Naturally one might say "Equivalent? Ridiculous!" What about the horizon line, and dawn and sunset times? But it is not necessary that the dial be equivalent in all respects: we need to build it as if it were in that theoretical equivalent place, and then 'slide it' parallel to itself to its intended location, adding the data of the place where we are. In the equivalent place, the substilar is vertical; while when we move the dial to where we are, the meridian line becomes vertical.

Therefore the equivalent place (for a vertical dial corresponding to what we are building) is characterised by two parameters that we always have, and we use to build the dial materially: a latitude equal to the complement of the substilar angle, and a difference of longitude, equal to the hour angle between the meridian line and the substilar line. If the dial also inclines in addition to declining, the problem of the equivalent place is more complicated, because the inclination is added to the mix.

I was forgetting an essential annotation: the equivalent dial is sought, in the conviction, perhaps correct and perhaps not, that its construction is easier and immediate. Otherwise it would be pointless to find it.

The formulas contemplate finding longitude and equivalent latitude, without design of the dial. So one prepares them, builds the easier dial, and then transfers it. It has been seen that perhaps finding the horizontal equivalent dial is more comfortable; but nothing prevents us from sliding the horizontal dial 90° on the same meridian, and transforming it into a vertical, which is also equivalent. Everything depends on personal preferences. And we then find the horizontal one equivalent.

To facilitate the reasoning, we must imagine a number of planes as they all are in the center of the sphere (the celestial sphere or the Earth - no matter): the equator, our horizon, and the plane of the dial. The North Pole is the pole of the equatorial plane; the Zenith is the pole of our horizon; and the perpendicular to the plane of the dial will find its pole at E. Also the opposite poles exist, but we ignore them.

If we imagine the dial moves to point E, always staying parallel to itself, it will emerge from the Earth and be tangent to the sphere at E; this is the horizontal dial in E: the equivalent dial!

All this talk is to show that a spherical triangle is obtained (NZE in the left half of Figure 16bis), of which three elements are known: NZ, the complement of the latitude; ZE, the complement of the inclination; and the NZE angle, the supplement of the declination (usually measured between the direction South and the perpendicular to the straight horizon line traced on the plane, that is AZE). If we solve it, EN will be the equivalent latitude, and the ZNE angle will be the difference of longitude, that is the equivalent longitude.

But I prefer the Analemma of Ptolemy to spherical trigonometry: I play with the compasses (Giordano Bruno did it also). Let us consider the example of Figure 16, to fix the ideas; it becomes more easy to make sense of: latitude 40°; EAST declination 35°, inclination 15°.

With reference to the left globe (not in scale!), NZ will be equal to 90°- 40°= 50°; the point E will surely be on the AEA' circle of Z pole, whose ZA = ZA' arc is in our case 90°- 15°= 75°, the complement of the inclination.

It now remains to trace the NZE angle: if however we refer to the plane of AEA' circle, it is easy to trace on such planes the desired angle: it will be A'CE= 180°- 35°. We have found on the globe the point E, whose coordinates are what we sought.

And from this point on, we consider the figure (fig. 16bis) to the right.

Fig. 16

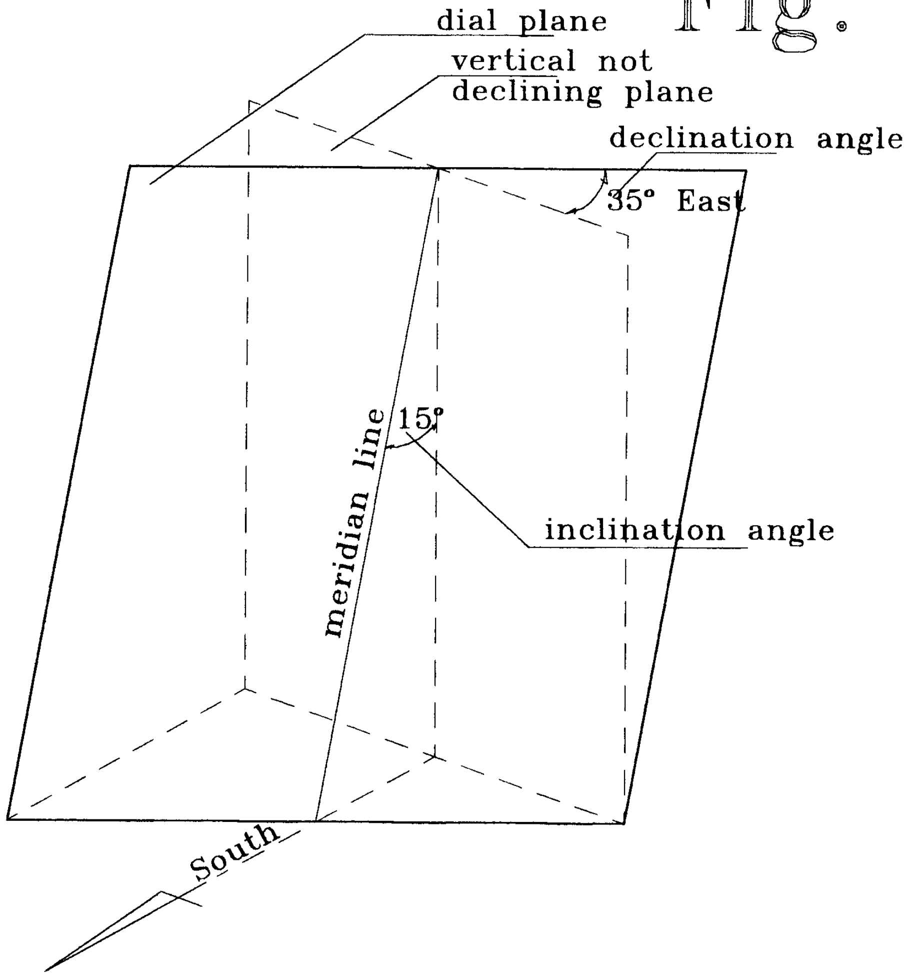
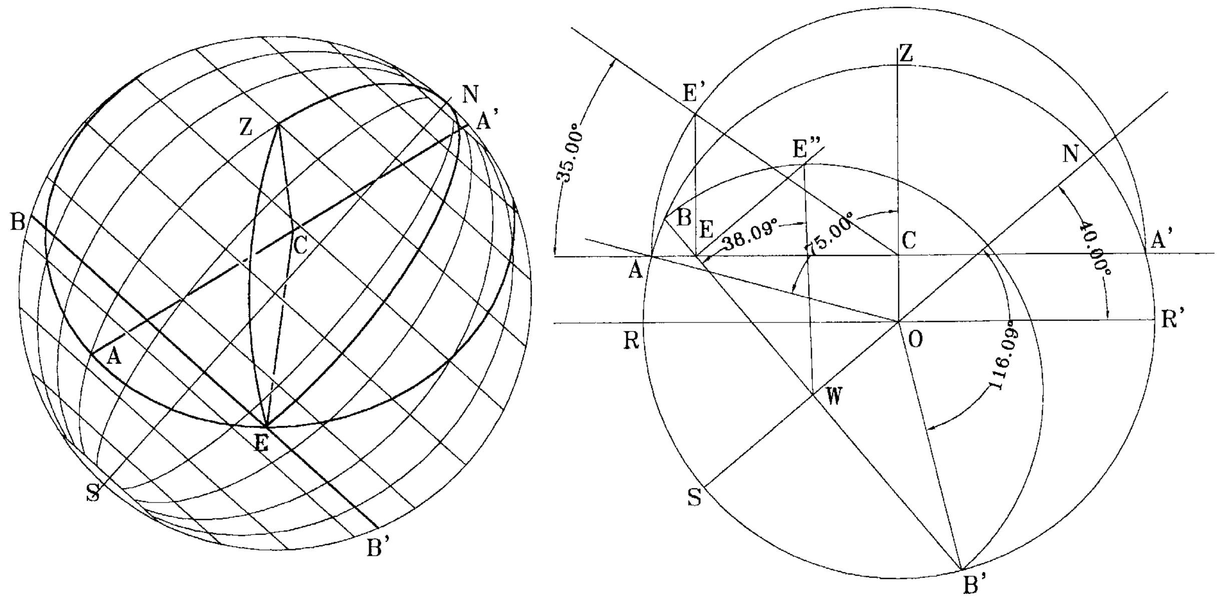


Fig. 16 bis

THE DIAL ON THE PLANE OF THE FIG. 16
WANTING THE EQUIVALENT HORIZONTAL DIAL



ANSWER: EQUIVALENT LATITUDE = 116.09° = 26.09° South
LONGITUDE : 38.09 East

* The SAZN circle is the sphere whose poles are N and S; the projection of the earth (or of the eighth sky of Ptolemy) is dealt with on the meridian plane; the center O is also the representation of the East and West points.

* ROR' is our horizon, transferred to the center of the sphere, whose pole is Z, the Zenith. The NOZ angle corresponds to the NZ arc of the other figure, 50°.

* If from Z we trace the ZOA angle = 75° (the complement of the inclination) the point A is obtained: the ACA' straight line is the trace of the circle, that in the other sketch is AEA,' which can be seen in pseudo-perspective. To find A we must capsize this circle, and do it on the same figure (This behaviour is not because we are stingy, or wish to save space – it simply is faster). On the capsized circle, that is on the AE'A' semicircle, mark ACE'= 35°, or better, its supplement A'CE'= 180°- 35°= 145°. Projecting E' on the straight line AA' yields the point E.

* If the circle that represents the sphere (SAZN) were entirely represented, with meridians and parallels, I could directly read the coordinates of the E point, but with scarce quality of the result.

* But, if I trace a perpendicular to the polar straight line, I get the trace of the parallel BWB', passing through E: the reader will compare it with the analogous parallel of the other figure. Capsizing the parallel, I find the point E": the BWE" angle (38.09°) is the equivalent longitude, thanks to the CAD.

* the B'ON angle = 116.09° corresponds to the arc NE of the other figure, and it is the equivalent latitude.

If then one prefers to design a vertical dial, its substilar angle will be 116.08° - 90°= 26.08°, corresponding to a latitude of 63.92°. That dial, given its position close to the polar circle, is not recommended – it would be better to come back to the horizontal dial.

I limited the determination of the angles to the hundredths of a degree, very sufficient for a dialist; but the graphic computerised calculus allows one to determine them with the same precision as the analytical methods.

I must finally, for moral honesty, add a note: the idea of applying the Analemmas came to me from reading a French text of 1637: Cursus Mathematici, by Pierre Hérigone. He does not draw at all the specific problem, but he introduces (in a dark and hardly comprehensible way) the concept of "pole of the surface of the dial," with the purpose of solving the problem of the general dial in one way, if possible independent of the place in which the dial is located.

The Circular Dial Of Oronce Finé

Attributing this dial to Oronce Finé is perhaps rash; there are precedents, connected (year 1075) with the famous "Saphea" of Azrachel (Azarquiel/ Abuyzhae), and with the Libros del Saber de Astronomia of Alfonso X el Sabio. The same kind of dial is illustrated by Fineus and by Apianus at the beginning of the 16th Century. It appears then in other texts of the same century.

Its advantage lies in its "universality." A reduced version of the Saphea is dealt with, with the addition of an indicator giving the altitude of the Sun; this allows one to read the hour in any day of the year and in any latitude. But it is also an Analemma.

What I write here is a personal adaptation of an essay published by G. Paltrinieri in the magazine "ANALEMA" of the Spanish dialists.

The THEORY

Figure 17/1 shows the Analemma, in which only the diurnal part is highlighted; I have traced the manaeus and also the so-called sciaterre (or "Zodiacal Radius" according to the ancient Italian texts). (The final state will be Figure 17/3).

On the diagram the hour lines are built; they could be built through points (the diagram of the italic lines could also be traced, but then the dial becomes practically illegible; or two dials would be needed, one for the morning hours and one for the afternoon).

Here we deal with arcs of ellipses, all convergent at the poles; but in practice, if we find the points on the equinox line (Figure 3bis illustrates the method) and on the two tropic lines, we could trace arcs of circles.

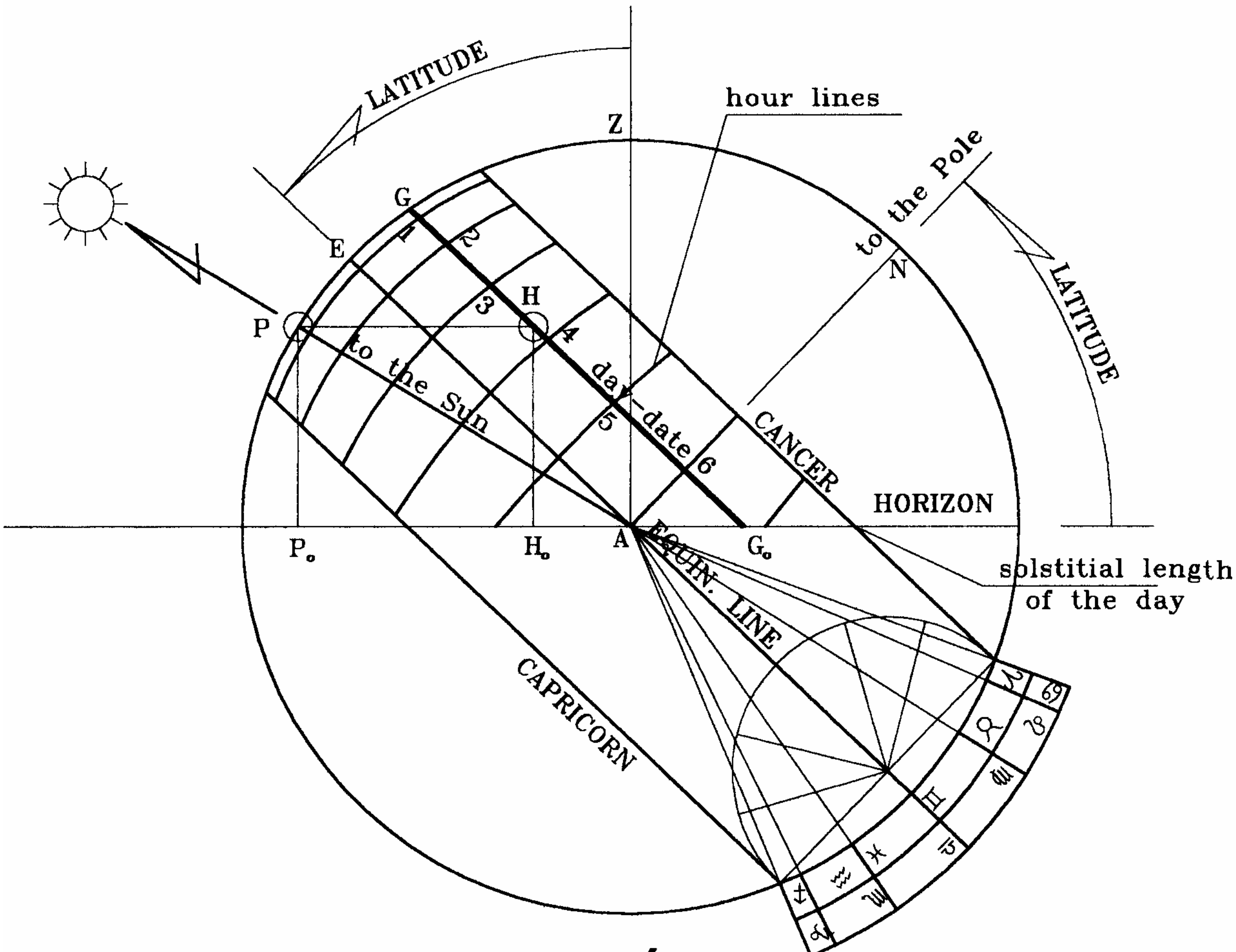


Fig. 17/1

The principle on which the dial is based is the following:

The inclination of the diagram is regulated by the position of the N point, that must be elevated above the horizon by an angle equal to the Latitude of the place. In the figure, the angle of latitude is also along the ZE arc, that is from the Zenith to the celestial equator. Contemplating the sun on a certain day and at a given hour, we find the point P on the edge of the Analemma; the height of the Sun will therefore be PP_0. If we move the height found in HH_0, on the parallel of the date we find the H point that allows us to read the hour. Obviously the H point could correspond to a morning hour or to an afternoon one. This could result in ambiguous readings near noon, when it is difficult to know if the height is before or after the noon point; it is however a "defect" common to all the altitude dials.

The "practical" disposition of the dial is illustrated by Figure 17/2: since we must use a plumb - line, the diagram is rotated through 90°, because the function of "local horizon" becomes the vertical line of the place.

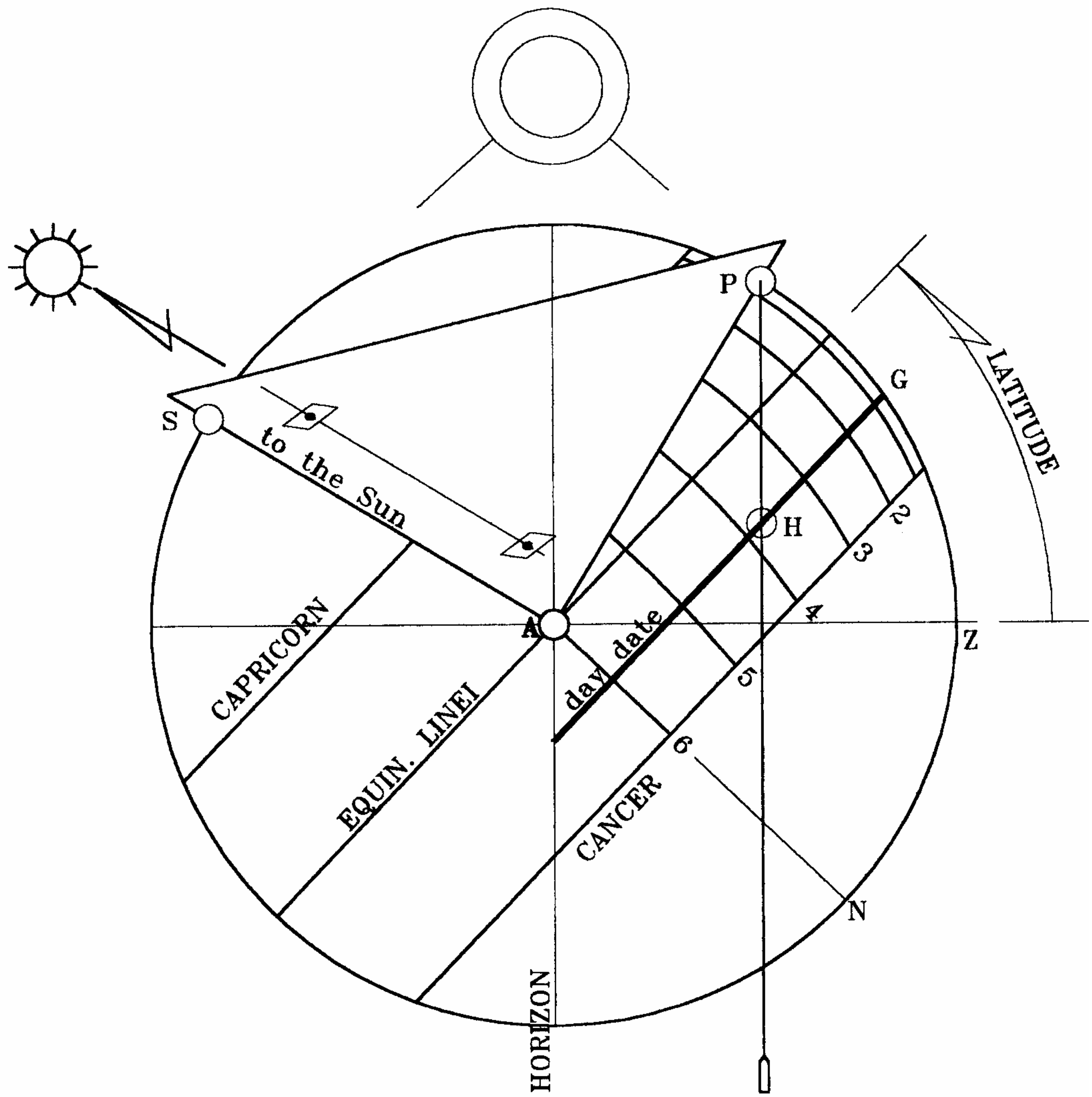


Fig. 17/2

We need therefore to superimpose on the diagram an alidade with the form of a right-angled triangle: You look to the Sun in direction OS, where two “pinnules” are inserted, and the plumb-line, attached to the extreme of the other leg of the triangle, is the PH straight line, parallel to the "new" local horizon.

In the H point read the hour, with evident coherence with the Fig. 17/1.

The Practice

To get a "universal" dial, we need to build the tool as a small astrolabe (in fact it is a Saphea of reduced size).

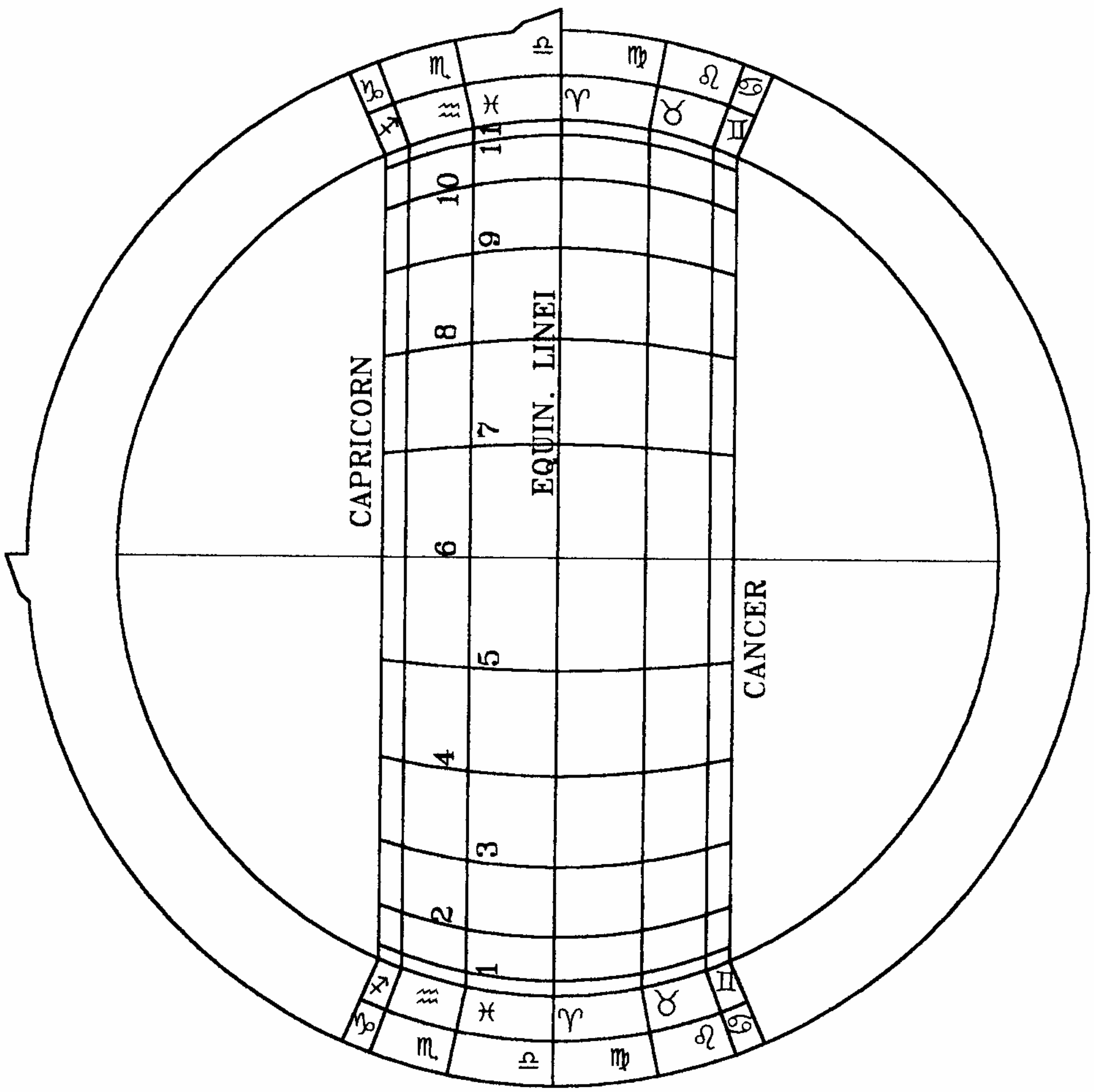


Fig. 17/3

The DISK or timpan (*plate*, in english 13th century texts) (figure 17/3). In Figure 17/3 the "true form" of the quadrant is illustrated, that makes it universal, for any latitude. Instead of the Signs it is better to insert the names of the months, and make the number of the parallels denser. The "notch" is useful for rotating it according to the latitude above the disk of the "Mother" (mater). The substantial equality of the Figure 17/3 with the Figure 17/1 is evident.

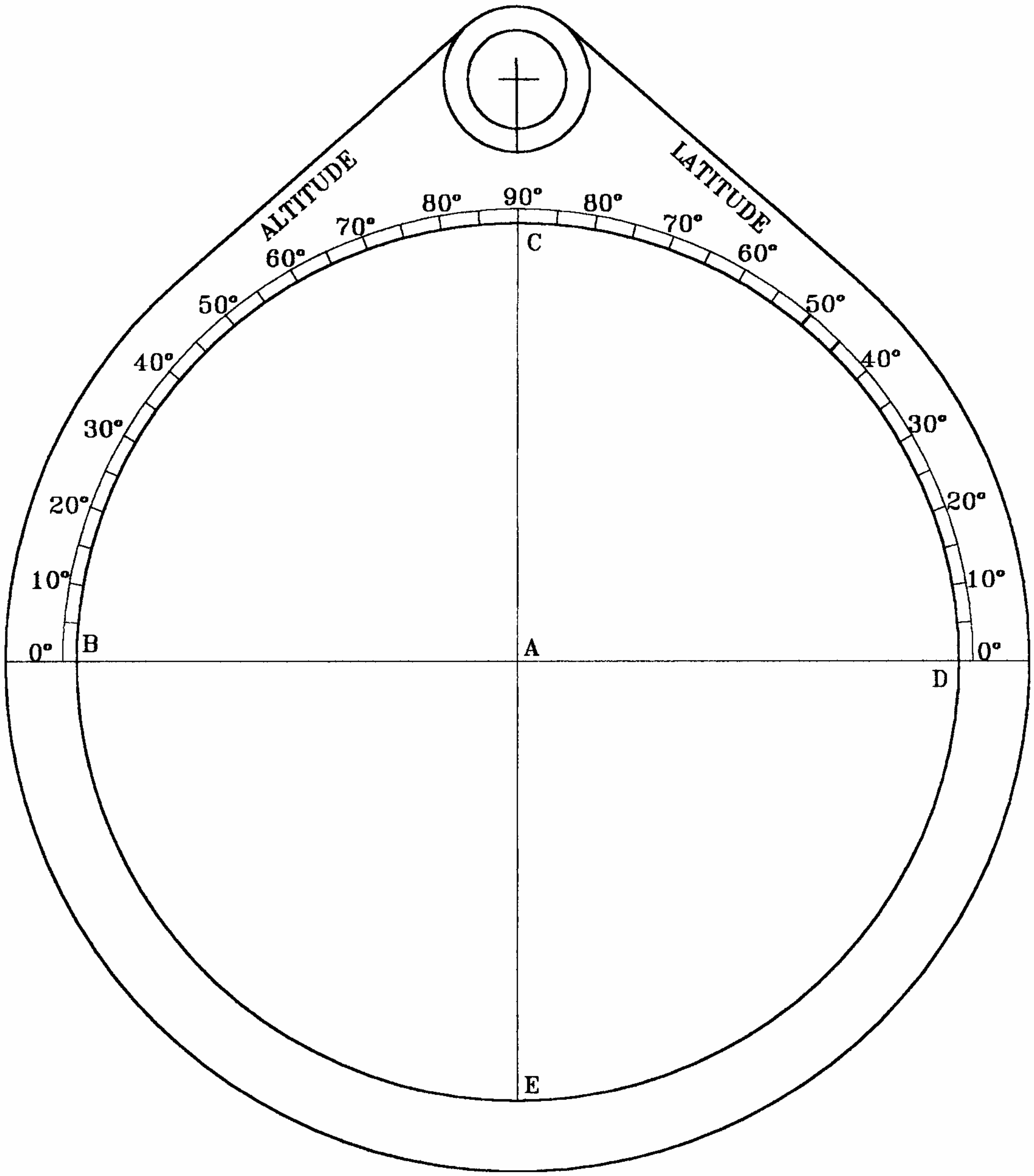


Fig. 17/4

The MOTHER (Figure 17/4): A disk on which a goniometer has been designed, for the indication of the latitudes. The ring should serve to hold the instrument suspended, so that the CE line is rigorously vertical. In practice the mother should be heavy enough to maintain this verticality when suspended below the ring.

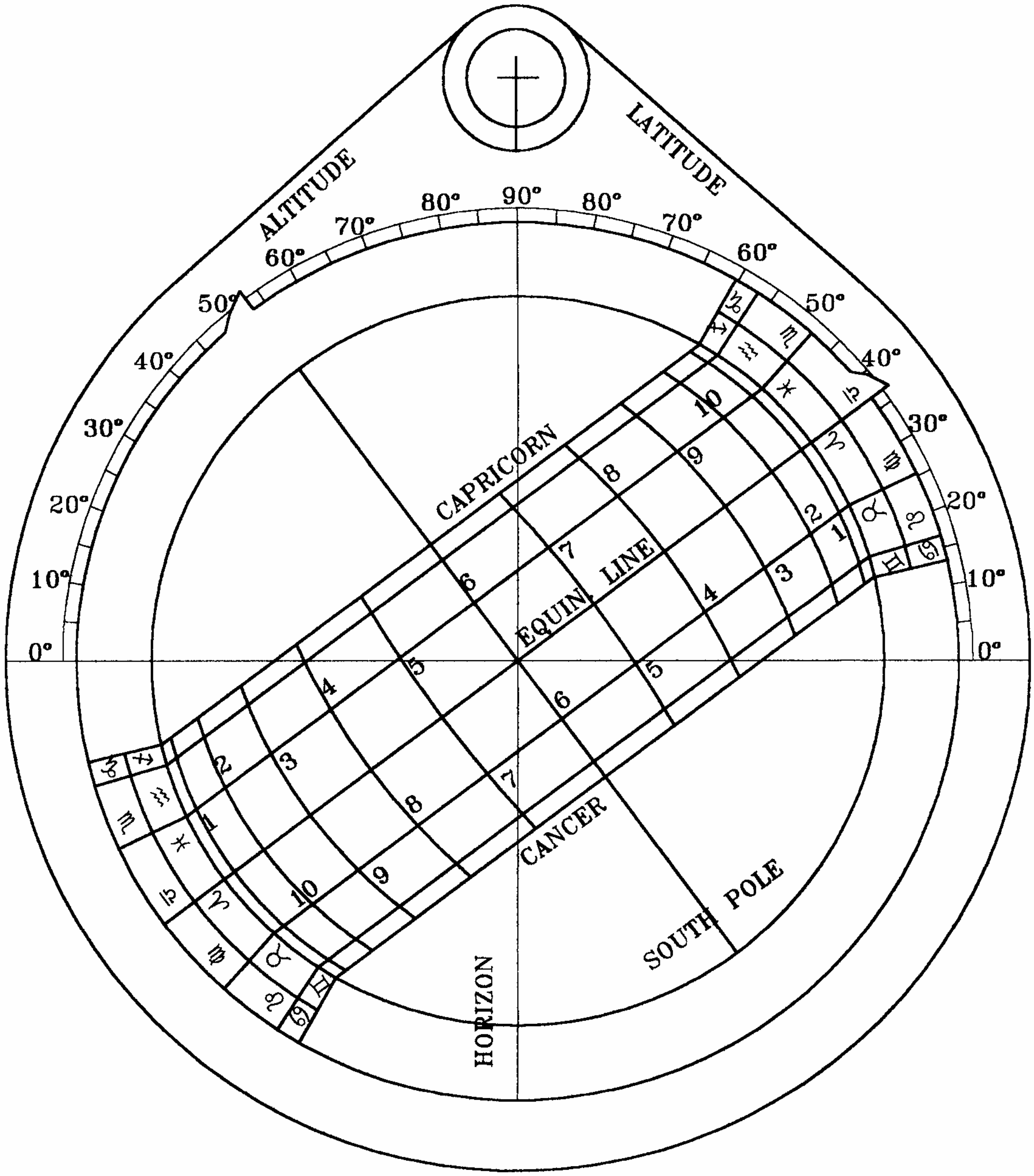


Fig. 17/5

Figure 17/5 illustrates how the timpan is affixed to the Mother. Thanks to the notch in the timpan, on the two goniometers the latitude and the height of the equatorial plane above the local horizon can be read.

THE TRIANGLE (Figure 17/6):

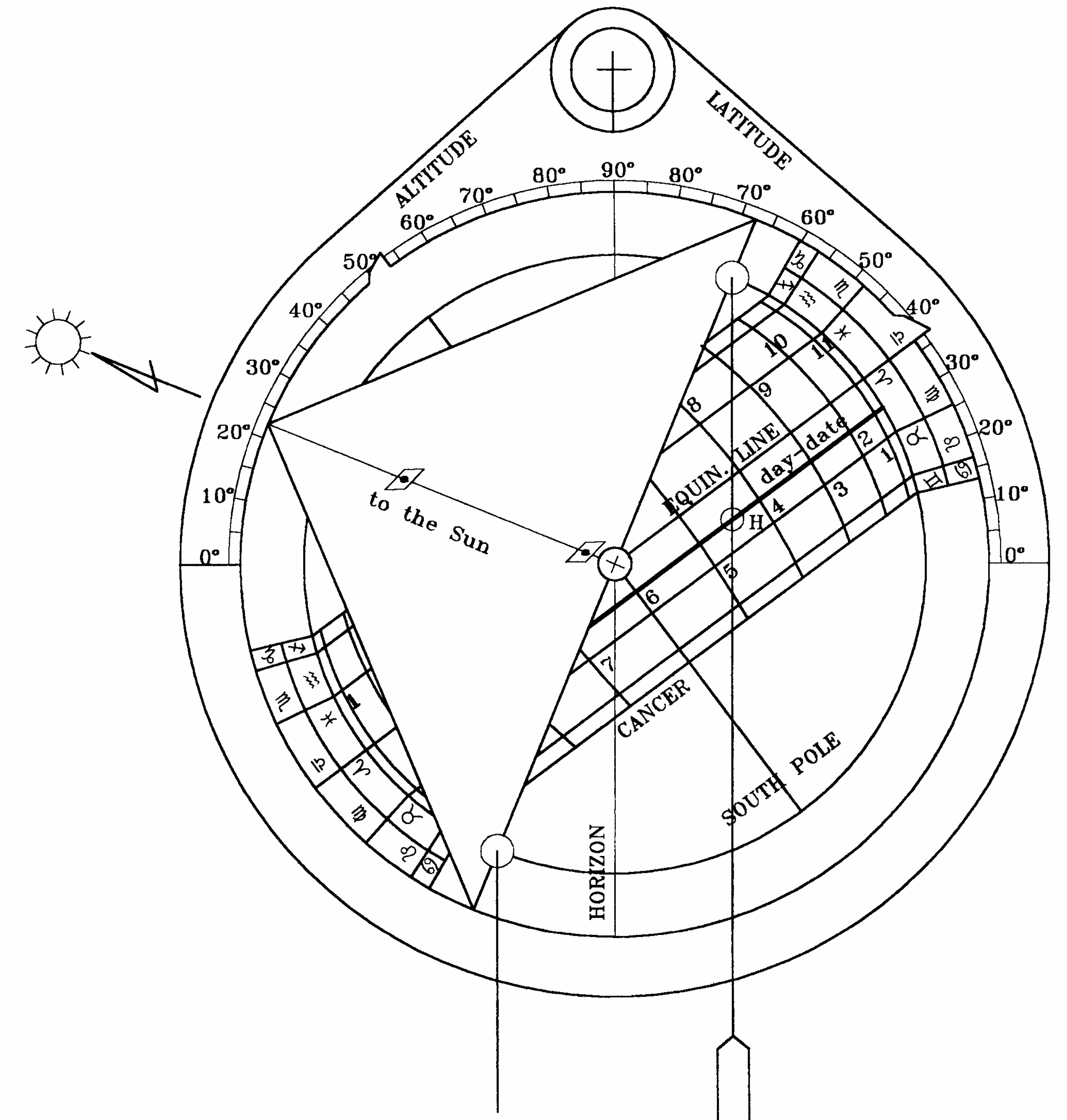


Fig. 17/6

The last detail that comprises the tool is the right-angled triangle (Alidade) shown in the figure. It is substantially the same as that in Figure 17/2, but some changes have been made for practical reasons: with the former layout, the plumb-line that allows us to read the hours would make the triangle unstable, so the orientation to the sun of the *pinnules* would be very difficult. This drawback is obviated be setting a second plumb line symmetrical to the first; that makes it necessary to build a triangle a little bit different from that of Figure 17/2; but the principle is identical.

Some notes:

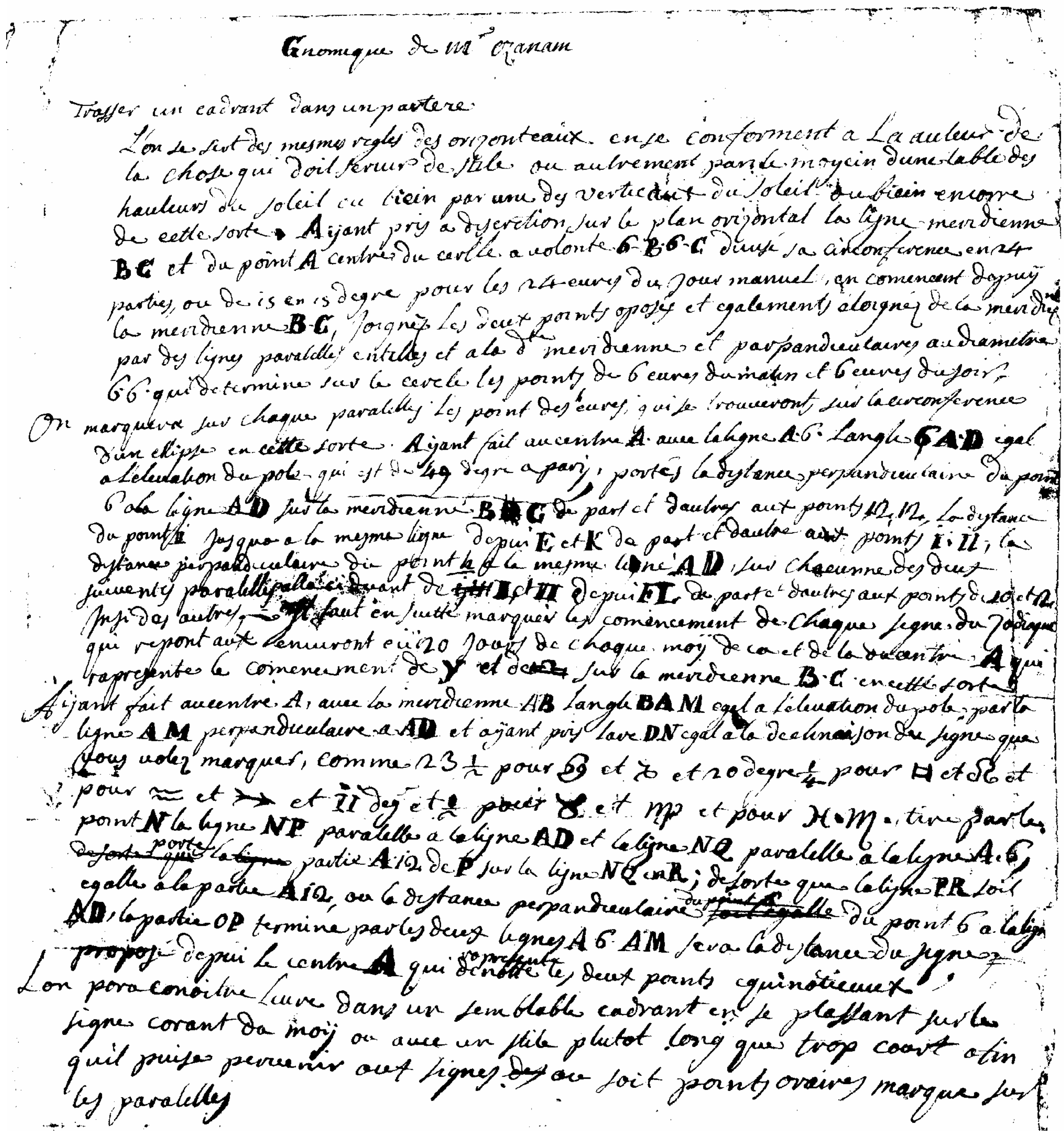
The tip of the triangle of Figure 17/6 allows us to read the height of the sun at the moment of the observation.

Arranging the triangle with the vertical hypotenuse (subjected to the plumb line), the lengths of the diurnal arcs of every parallel line could be read for the latitude marked on the Mother. Therefore it is possible to calculate the italic hour from the data furnished by the dial.

Why Is The Analemmatic Dial Called 'Analemmatic'?

The question of why the analemmatic sundial is called 'analemmatic' is fully justified, because it is difficult to see the Analemma of Vitruvius in the dial universally called by that name, a name which we inherited from our predecessors of almost 400 years ago without knowing why or asking why.

Beginning to think about the problem, I present to the reader the reproduction of the explanation I have found in a paper of unknown origin, probably of the 18th century, found in a junk shop, written in French.



Gnomique de Mr Ozanam

Trasser un cadrant dans un partere

L'on se sert des mesmes regles des orizonteaux en se conforment a La auteur de la chose qui doit servir de stile ou autrement par le moyen dune table des hauteurs du soleil ou bien par une des verticaux du soleil ou bien encore de cette sorte. Aijant pris a discretion sur le plan orizontal la ligne meridienne BC et du point A centre du cercle a volonte 6 B 6 C divise sa circonference en 24 parties ou de 15 en 15 degre pour les 24 heures du jour manuel, en comencant depuis la meridienne BC, joignes les deux points opposes et egalement eloignes de la meridienne par des lignes paralelles entre elles et a la dte meridienne et perpandiculaires au diametre 66 qui determine sur le cercle les points de 6 heures du matin et 6 heures du soir.

On marquera sur chaque paralelles les point des heures qui se trouveront sur la circonference d'un ellipse en cette sorte. Aijant fait au centre A avec la ligne A6 Langle 6AD egal a l'elevation du pole qui est de 49 degre a paris, portes la distance perpandiculaire du point 6 a la ligne AD sur la meridienne BAC de part et d'autre aux points 12, 12, la distance du point E jusqu'a la mesme ligne depuis E et K de part et d'autre aux points I, II, la distance perpandiculaire du point a la mesme ligne AD sur chacune des deux suivantes paralelles [illegible] de I et II depuis F L de part et d'autre aux points de 10 et 2 ainsi des autres. Il faut en suitte marquer les comencement de chaque signe du zodiaque qui respont aux environs du 20 jours de chaque mois de ca et de la du centre A qui represente le comencement de ♈ et de ♎ sur la meridienne BC en cette sorte.

Aijant fait au centre A avec la meridienne AB langle BAM egal a l'elevation du pole, par la ligne AM perpandiculaire a AD et aijant pris l'arc DN egal a la declinaison du signe que vous voules marquer, comme 23 1/2 pour ♋ et ♑ et 20 degre 1/4 pour ♊ et ♌ et pour ♒ et ♐ et 11 deg et [illegible] pour ♉ et ♍ et pour ♓ ♏, tires par le point N la ligne NP paralelle a la ligne AD et la ligne NQ paralelle a la ligne A6, ~~de sorte que la ligne~~ portes la partie A12 de P sur la ligne NQ en R; de sorte que la ligne PR soit ~~parallelle~~ AD, la partie OP termine par les deux lignes A6 AM sera la distance du signe ~~proposé~~ depuis le centre A qui represente les deux points equinotieux.

L'on pora conoitre l'eure dans un semblable cadrant en se plassant sur la signe corant du moy ou avec un stile plutot long que trop court afin quil puisse pervenir aux signes ou soit points oraires marque sur les paralelles

But this is a paraphrase of what can be found in the VIIIth problem, 2nd volume of Ozanam's <u>Récréations mathématiques</u>, and therefore we might do well to reconsider the original text:

Trace from the point A on the horizontal plane the meridian line BC, and lay out, always from the same point A, the circle 6B6C: divide its circumference in 24 equal parts (each of 15 degrees), the hours of the natural day, starting from BC, the meridian line.

Join the opposite points, equally distant from the BC line, with lines that will be therefore parallel to each other and to the meridian line BC, and perpendicular to the diameter 6-6, (diameter that joins the 6 morning point with the 6 evening point on the circle).

On each such parallel we will find the points of the hours, that will be on the perimeter of an Ellipse, applying the following method: Trace on the center A, beginning from the line A6, the angle 6AD equal to the elevation of the pole, in Paris 49°, and transfer the perpendicular to AD (traced beginning from the point 6), on the BC meridian line, on both sides from center A, finding the points 12, 12; then the length of the perpendicular, from point I to AD, on each of the two parallels nearer to the meridian line, beginning from points E and K, on both sides, finding the points 1 and 11; the perpendicular to AD from the point H must be transferred on the following parallel, nearest to the preceding, beginning from F and L, always on both sides, finding 2 and 10 points; and so on for the remaining.

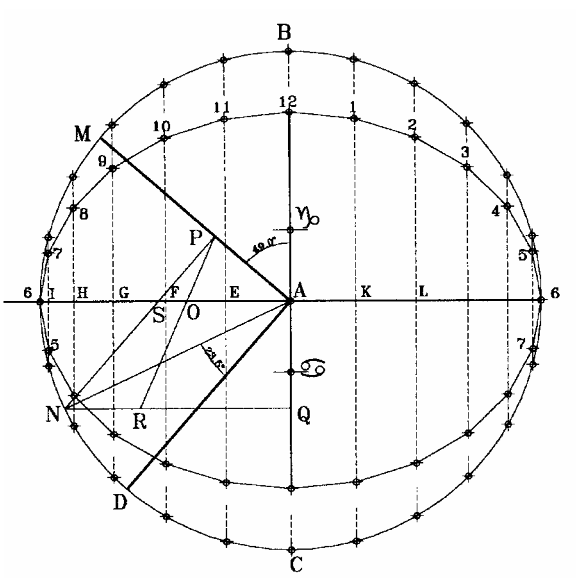


Figure 1. The constuction following Ozanam

Subsequently we need to mark the beginning points of every sign of the Zodiac, corresponding approximately to the 20th day of each month, along the meridian line, on the two sides of center A that corresponds to the beginnings of ♈, and ♎.

Trace on the center A, with reference to the meridian line, angle BAM equal to the elevation of the pole (AM to be perpendicular to AD), and take the arc DN, equal to the declination of the sign to be characterized (that is 23°30' for ♋ and ♑; 20° 15' for ♊ ♌ and for ♒ ♐; 11°30' for ♉ ♍ and for ♓ ♏), then, beginning from N, trace the line NP parallel to AD, and NQ parallel to A6.

Transfer the distance A12 from P to point R on NQ, so that POR is as long as A12, (that is to the length of the perpendicular from 6 to AD). The OP part that stays between A6 and AM will be the distance looked for of the sign, to be laid off beginning from the center point A, that serves as the point of the equinoxes.

Once we have traced the dial with its ornaments, the hours will be found on it through the rays of the Sun, as in other dials, if one stands approximately on the degree of the current sign of the Sun, with this difference: in the normal horizontal Dial the stylus must have a precise dimension,

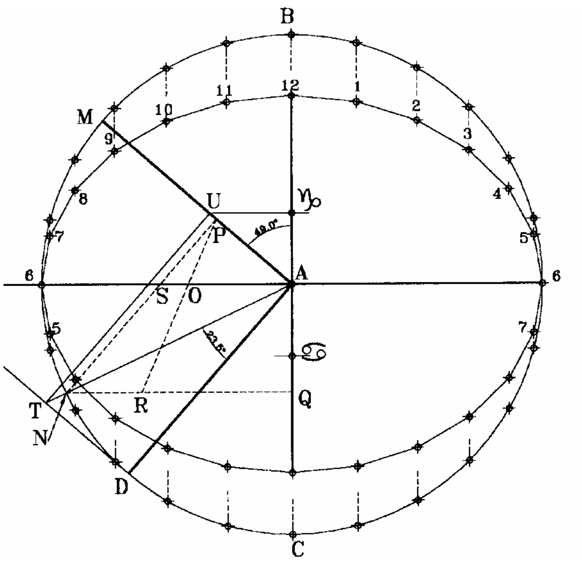


Figure 2. A more 'accessible' variant

while here it could be of any greatness. But it should be a little bit long, because if it is small, the summer shadow could be so short as to not reach the hour points marked on the parallel lines, and so it could not point out the hours.

The text of the leaflet was rather confusing, and only recourse to the text and to the original sketch of Ozanam (Figure 1) has allowed me to understand at least the sequence of the operations.

In substance, once the points on the ellipse (and Ozanam does not explain why it is an ellipse) have been found, a series of graphic developments defines for each declination δ a point N, so that the arc DN is equal to the declination: then one needs to do the construction pointed out in the figure and explained by the text to find PO and transfer its length onto the meridian line, thus finding the Zodiac point desired (in the sketch I traced only the solstice: PO= A♋ and A♑, not more).

If we play with the trigonometry, taking into account the similitude between triangles NRP and SOP, we discover that the distance looked for is (tg δ cos φ), but Ozanam does not dream of explaining why that quantity and not another.

I have also looked at Bédos de Celles: his solution is graphically different, but substantially identical to the illustrated one; and not even Bédos furnishes information on the rationale.

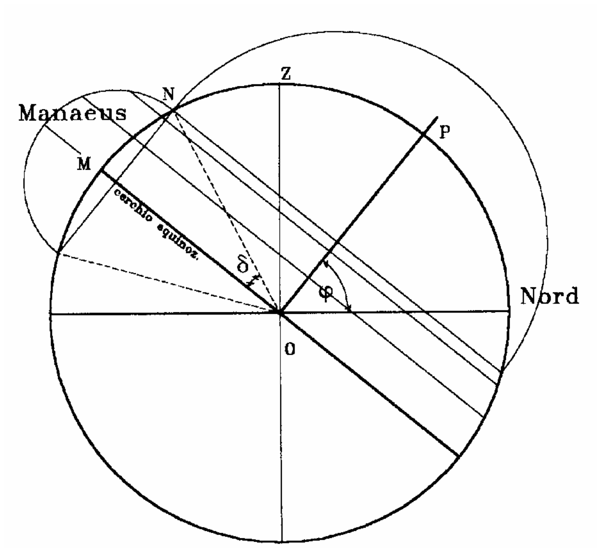


FIGURA A

I also have looked for a figure that realizes (tg δ cos φ) without sophisticated turns (Figure 2): tracing the right-angled triangle ADT, U could be found, and accordingly the A♑ distance directly; it deals with a pair of right-angled triangles that I do not believe require further explanations (we could find AU = tg δ tracing the angle of declination with vertex in D directly, instead of in A, simplifying the figure, but I here wanted to give "continuity" with the sketch of Ozanam. And… it then is useful for what you will find below).

The fact remains that all this does not explain in any way how the term 'analemmatic' was born.

Naturally I did not surrender, and I have looked for what could be the “iter” coming from the direct application of the Analemma of Vitruvius, finding any developments rather indicative, or at least of interest for whoever has interest in this matter.

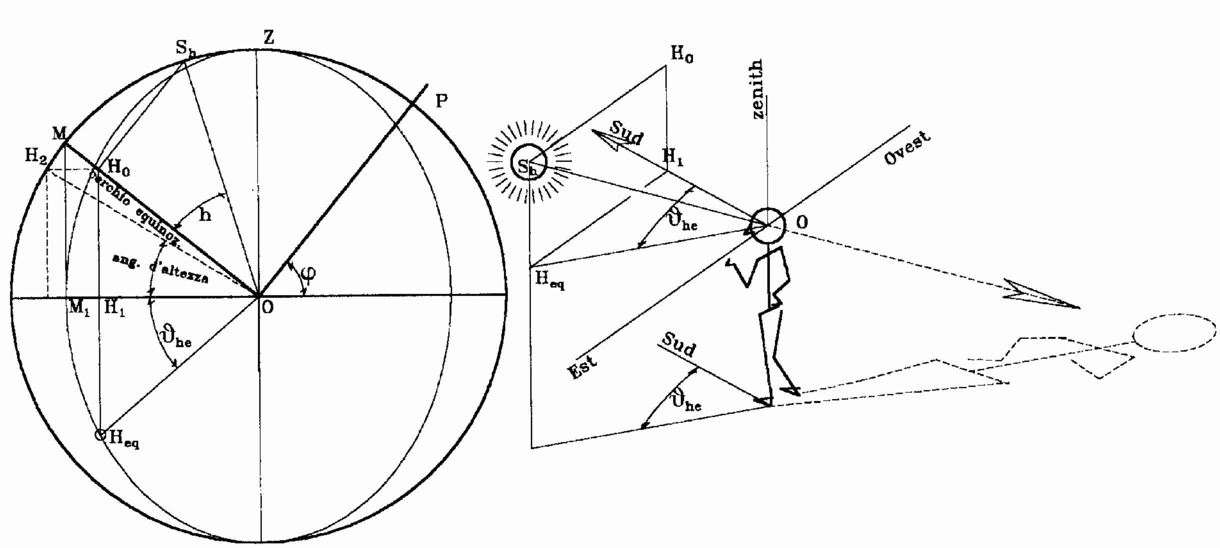


FIGURA B

Put it in these terms: at the epoch of Vaulezard (who is said to be the first builder), the Analemma was still one of the tools used to trace dials.

Ozanam, on his side, has quoted a particular application of the Analemma sort from the Jesuit De St. Rigaud (*the quotation is debatable: from the type of illustrated Dial we could go back to Apianus, around 1520*), that has some relationship with what I write here. However, then Ozanam has not used this particularity to illustrate the analemmatic. Perhaps because he has not illustrated it at all.

Therefore, to justify the name of the dial, I have found it logical to imagine putting myself in place of Vaulezard: a leap of imagination, in short, for which I do not feel up to affirming that the origin of the name comes from something similar to the development of the following demonstrations.

If then anyone does not believe: patience. It will mean that I added a page to the by now broadly unknown or forgotten "theory of the Analemma", conscious that someone probably did it at least 400 years before.

In Figure A, I have illustrated the Analemma, with only the declination parallels of the entries into the summer signs. The construction is known, so I do not dwell upon it.

Figure B illustrates how the Azimuth of the Sun could be found on the equinox day, at the h hour of the morning. Let the Equator circle be capsized around the MO line and on the turnover find the S_h point, corresponding to the h hour angle; project S_h on the MO trace of the equator, finding H_0; the projection H_0H_1, duly brought again on the maximum circle in H_2H_3, is the parameter of the height of the Sun to the h hour.

If, from H1 we turn back the distance $H_1H_{eq} = S_hH_0$, the angle H_1OH_{eq} is the Azimuth θ_{he} of the Sun in the S_h position.

H_{eq} points corresponding to every hour are certainly on an ellipse, and that is evident from the figure, as the M mediumcoeli point of the equator is projected in M_1, the H hour points in H_{eq}, while the 6 hours points do not suffer projection. The ellypse has OZ and OM_1 as principal rays (where Z must be read as the turnover of the West point of the local horizon plane). I added a little stick figure in pseudo-perspective also; I think it illustrates in a less complex way the joke. I add that the ellipse has a form depending on the latitude, and is "squeezed" more and more as we near the equator.

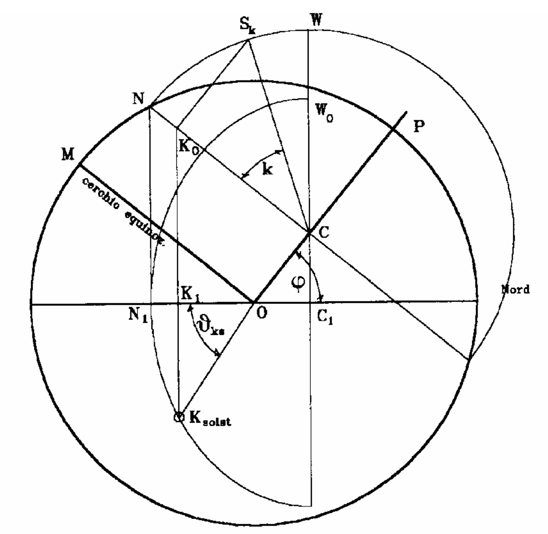


FIGURA C

C Figure

I repeated the same operations also for the Sun in the parallel of solstice: I have pointed out the hour angle with k, but the reader is begged to consider it correspondent to the same h hour of the preceding figure. Also in this figure I traced the ellipse: it however has its center in point C_1, and its diameters are proportional to the ray circle CN, that it is the parallel of declination. That means that, the latitude being unchanged, the ellipse of Figure C and that of Figure B are similar, and also the distances C_1K_1 and K_1K_{solst} are proportional respectively to the distances OH_1 and H_1H_{eq} of the B figure, in the same relationship of proportion.

So, at the same hour, the Azimuth θ_{ks} is larger than at the equinox, because the point O in which the observer stays is not the center of the ellipse.

D Figure

With the purpose of magnifying the ellipse of Fig. C, without changing the angles important to our puposes (Azimuth and Height) I imagined projecting the spherical surface from center O, on a cylindrical surface, tangent to the sphere along the equator. The mediumcoeli solstice point on the sphere, named N_{sf}, is projected onto the point of the cylinder that I named N_{cil}, and the projection of the declination parallel will be the circle outlined with C_{cil} center, and $C_{cil}N_{cil}$ radius. The figure has also the projections of other declination parallels, but I only "hinted" at them, for I do not wish to make the sketch overly complex. (*Incidentally, since I dragged him into this discussion, I observe that this is the variation on the Analemma operated by St. Rigaud*)

Doing the accounts, OC_{1cil}= (tgδ cosφ). (*The same as in the figure of Ozanam!*).

E Figure

Repeating, I have superimposed the two lay-outs of Figs. B and C in the same figure: I observe that now $H_1O = K_1C_1$, because they are projections, respectively, of H_0O and K_0C, identical quantities. For the same reason also $H_1H_{eq} = K_1K_{solst}$.

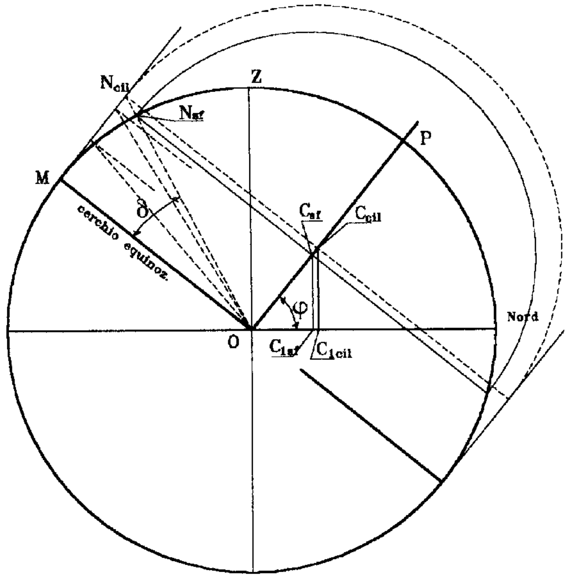


FIGURA D

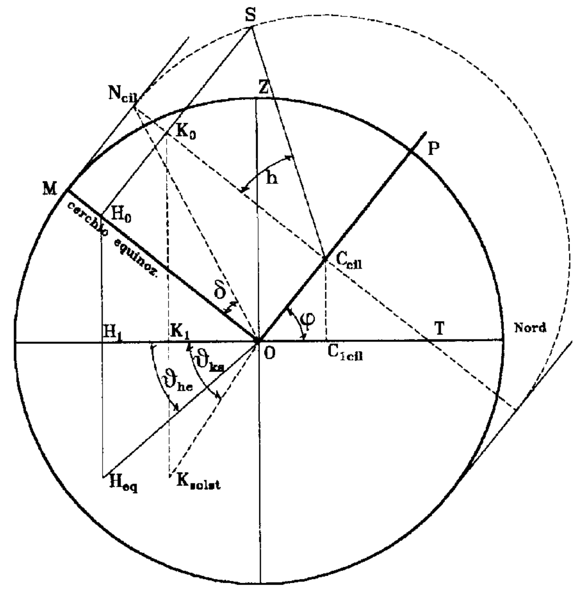


FIGURA E

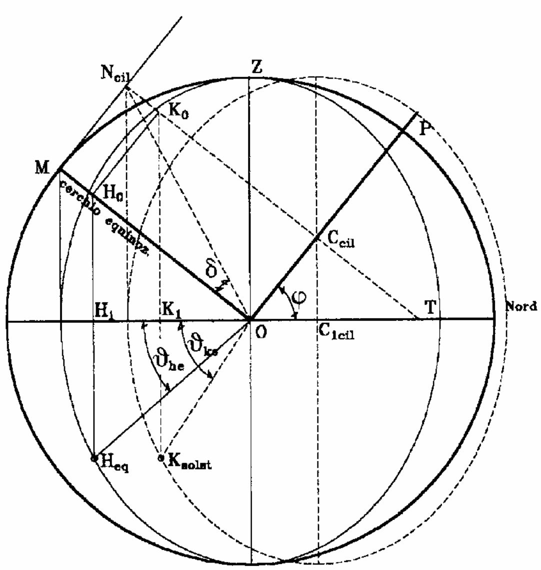


FIGURA F

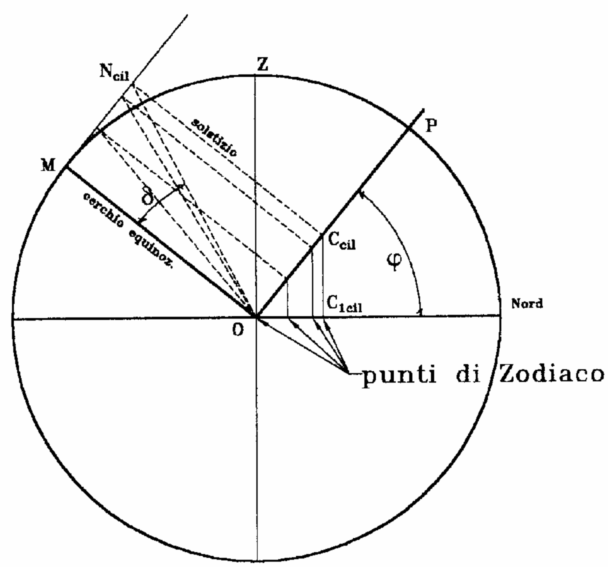


FIGURA G

F Figure

Here it has been underlined how the two ellipses are identical, and how the second ellipse is moved by the quantity OC_{1cil} as regards the equinoctial ellipse. If they come superimposed, H and K points (with respective indexes) would be superimposed on the horizon plane. It also proves, indirectly, what the shift of the gnomon must be in the solstice as regards its equinoctial position, justifying also the distance ($tg\ \delta \cos \varphi$) found in Ozanam.

Figure G is perhaps useless, but has been done only to show how to find the "points of the Zodiac" on the meridian line.

That is substantially identical to the "variant" to Ozanam I presented above in Fig. 2.

Will this be the reason for which the dial is named Analemmatic?

[Editor's Note: Ozanam's work went through very many editions between 1694 and 1844. The identification here as Problem 8 of the 2nd volume comes from the 1725 French edition, which consisted of 4 volumes with new material added by Grandin. After several more editions, the work was rewritten by J.E. Montucla in 1778, at which time the material on the analemmatic sundial was dropped.]

An Analemma,
Shewing by Inspection, the Time of Sun Rising and Sun Setting, the Lengths of Days and Nights, the Beginning and End of Twilight, and the Point of the Compass on which the Sun Rises and Sets, for every five Degrees of Latitude, and for every five Degrees of the Sun's North and South Declination. Which by Judging of the Intermediate Declinations corresponding to the Days of the Mont, and the intermediate Degrees of Latitude, answers for all Latitudes and throughout the Year.
S. Dunn, Teacher of the Mathematical Sciences (London, England, 1774)

This Delineation has four kinds of Lines. First, Straight Lines from one Side of the Circumference to the other, and passing through the Center, at the Distance of each five Degrees in the Circumference, these Lines represent the Horizons of Places in those respective Degrees of North and South Latitude. It is easy to Judge by the Eye for intermediate Degrees and Part of a Degree as accurately as the Delineation requires.

Secondly, Straight Lines parallel to the Equator, drawn to every five Degrees of North and South Declination opposite to which, in a Direction with those Lines, are the Months and Days of each Month, when the Sun has those respective Degrees of North and South Declination. To Inhabitants on the Earth, which are as far from the Equator as those Parallels of Declination, the Sun is vertical at Noon, on those respective Days of the Months. For intermediate Days of the Months, or for intermediate Degrees of the Sun's Declination, the Eye will be able to judge nearly by the Order of the Scale.

Thirdly, Hour-Lines are drawn at the Distance of each half Hour from the Center. And as 30 Minutes make half an Hour of Time, a tenth Part of one of these half Hours will be nearly Three Minutes of Time, and a fifth Part nearly Six Minutes of Time; this the Eye can judge of without intermediate Divisions, which would cause a Multiplicity of Lines, and thereby Confusion.

Fourthly, Circles are drawn round the Center at the Distance of each 11½ Degrees from each; these are to Shew on what Point of the Compass the Sun Rises and Sets, and on what Point of the Compass the Twilight begins and ends at all places of the Earth and Sea, and throughout the Year. When the Latitude and Declination are both North or both South, the Sun Rises before Six of Clock, between the East, and the Elevated Pole; otherwise between the East and the Depressed Pole.

Example. Suppose the Time of Sun Rising, Setting, Length of the Day, Night, the Time when the Twilight begins and ends and what Point of the Horizon the Sun Rises and Setts on, required for the Latitude of the Lizard Point in England or for Frankfort in Germany, or Abbeville in France, April 30th.

The Latitude of either of these Places by the Map is nearly 50 Degrees North. Guide your Eye from Latitude 50º in the Quadrant of North Latitude toward the Center until you come to the Parallel of the Sun's Declination opposite to April 30th and note where those two Straight Lines do intersect each other. From which Point Count among the Curved or Hour-lines, toward the

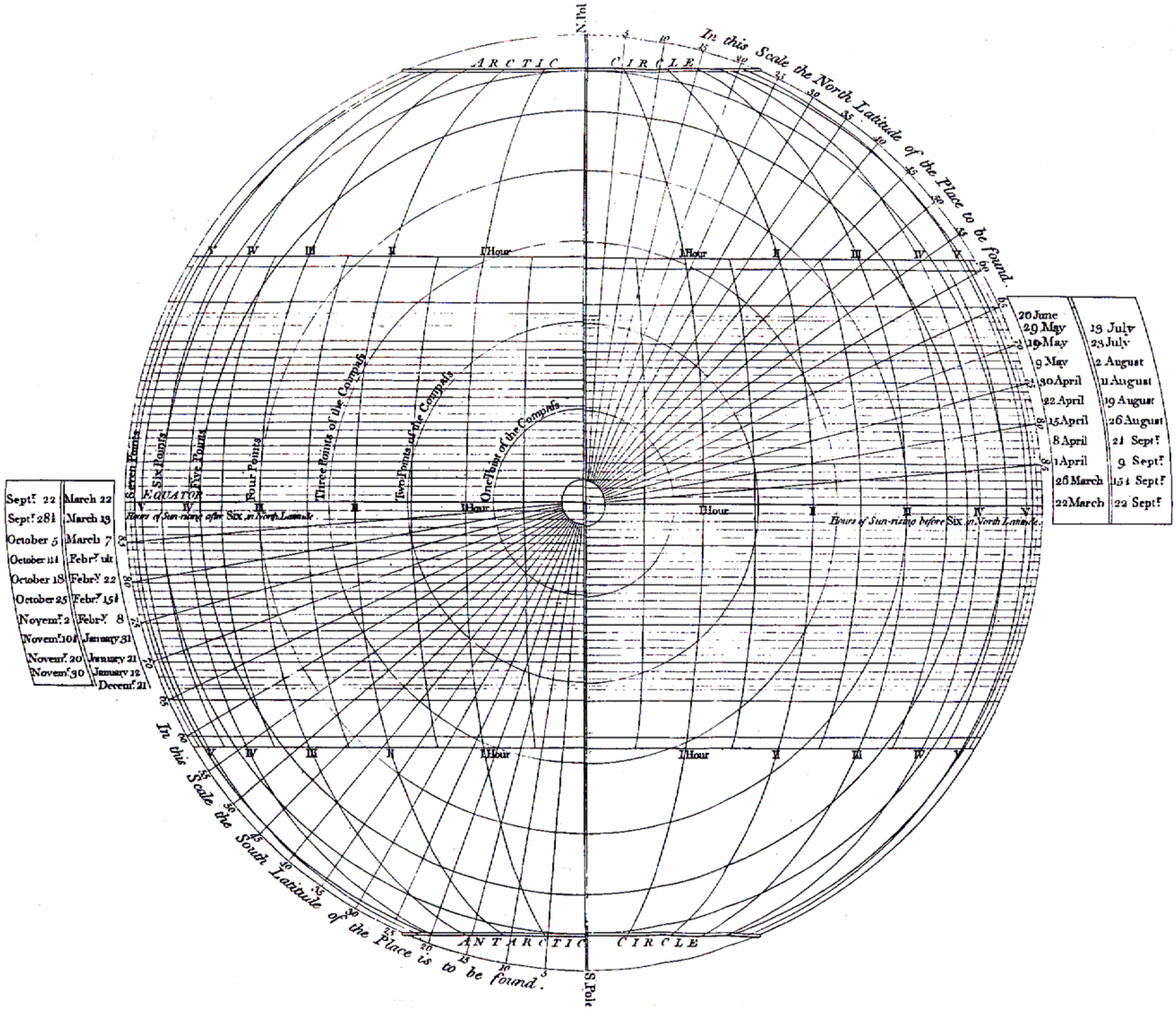
Center, and there will be One Hour and fifteen Minutes. So the Sun Rises 1h. 15' before Six, and Sets 1h. 15' after Six. The Sun Rises at 4h. 45' in the Morning, and Sets at 7h. 15' in the Evening; the Length of the Day is 14h. 30', and the Length of the Night is 9h. 30'.

With the length of 17½ Degrees of each Arch, move it along perpendicularly under the Horizon of 50º until it cuts the Parallel of 30º and from where these intersect Count among the Hours toward the Center, and it gives 3h. 46' the time of the Beginning of Twilight before Six in the Morning. So the Twilight begins at 2h. 14' Morning and ends at 9h. 46' Evening.

The Point where the Sun Rises is two and ⅛ of the Concentric Circles from the Center; therefore the Sun Rises 2⅛ Points from the East or N.E. by E. 7/8 E. and Sets N.W. by W. 7/8 W. The Point where the Sun is at the Beginning of Twilight, is five Concentric Circles from the Center nearly, therefore the Twilight begins at N.E. by N., and ends at N.W. by N.

N.B. The Twilight begins and ends when the Sun is 18 Degrees under the Horizon, which is nearly the Length of 17½ Degrees taken from the Arch as above.

London. Printed for Robt. Sayer. No. 53 in Fleet Street, as the Act directs. 10 January 1774.


N. Pol
In this Scale the North Latitude of the Place to be found.
ARCTIC CIRCLE
ANTARCTIC CIRCLE
S. Pole
In this Scale the South Latitude of the Place is to be found.
EQUATOR
Seven Points
Six Points
Five Points
Four Points
Three Points of the Compass
Two Points of the Compass
One Point of the Compass
Hours of Sun-rising after Six in North Latitude.
Hours of Sun-rising before Six in North Latitude.
I Hour
II
III
IV
V
20 June
29 May
19 May
9 May
30 April
22 April
15 April
8 April
1 April
26 March
22 March
13 July
23 July
2 August
11 August
19 August
26 August
2½ Sept.
9 Sept.
15½ Sept.
22 Sept.
Sept. 22
Sept. 28½
October 5
October 11½
October 18
October 25
Novem. 2
Novem. 10½
Novem. 20
Novem. 30
March 22
March 13
March 7
Febr. ult
Febr. 22
Febr. 15½
Febr. 8
January 31
January 21
January 12
Decem. 21